May 2009

Central European University

Advanced Time Series Analysis

2 Hours

Answer ANY THREE questions, out of 7.

1. This question concerns the properties of a stationary Gaussian sequence

$$x = \{X_t\}_{t=1}^{\infty} \ .$$

(a) What does it mean to say that x is Gaussian? Give a definition in terms of the finite dimensional distributions (fidis) of the sequence, and explain what this implies about the distribution of the sequence as a whole.

(b) Does the attribute ‘stationary’ here imply strict stationarity or wide-sense stationarity? Explain.

(c) Is x a linear process? Justify your answer.

(d) Is x always weakly dependent? If so, why? If not, can you specify some additional conditions that would ensure this property holds?

(e) Is x always ergodic?

2. Consider the stochastic processes generated by the difference equations

$$\Delta x_t = 0.2\Delta x_{t-1} - 0.4 x_{t-2} + \varepsilon_t \qquad (1)$$

$$\Delta x_t = 0.2\Delta x_{t-1} - 0.36 x_{t-2} + \varepsilon_t \qquad (2)$$

where $\varepsilon_t \sim iid(0,\sigma^2)$ and $\Delta = 1 - L$ is the difference operator.

a) Are the processes $\{x_t\}$ stationary, in each case? Are they weakly dependent? Explain your answers.

b) Discuss the characteristics of the Wold representations of these processes.

c) Do the processes $\{\Delta x_t\}$ also possess Wold representations?

d) Now consider the equation

$$\Delta x_t = 0.2\Delta x_{t-1} + \varepsilon_t \qquad (3)$$

Do the corresponding processes $\{x_t\}$ and $\{\Delta x_t\}$ have Wold representations? Explain.

3. Let $\{x_t, \mathcal{F}_t\}$ and $\{y_t, \mathcal{F}_t\}$ be martingale differences with respect to the same filtration.

(a) Which of the following sequences $\{z_t\}$ are martingale differences with respect to $\{\mathcal{F}_t\}$?

i) $z_t = x_t + y_t$; ii) $z_t = x_t y_t$; iii) $z_t = x_{t-1} y_t$; iv) $z_t = x_{t-1} + y_t$.

(b) Now assume that $\{x_t\}$ is not a martingale difference, although it is integrable and adapted to $\{\mathcal{F}_t\}$. How does this change your answers to i)-iv) in part (a)?

(c) Now, in addition to the conditions in part (b), further assume that x_t and y_t are contemporaneously independent, for each *t*. How does this change your answers to i)-iv) in part (a)?

(d) Briefly discuss the applications of the martingale concept in time series analysis..

4. Consider the ARMA(1,1) model

$$x_t = \lambda_1 x_{t-1} + \varepsilon_t + \theta_1 \varepsilon_{t-1},\ \varepsilon_t \sim \mathrm{NI}(0,\sigma^2)$$

(a) Describe how, given a sample $x_1,\ldots,x_T$, you would compute least squares estimates this model (point estimates and standard errors).

(b) Explain how, given your estimates in (a), you would compute forecasts and forecast standard errors for $x_{T+1},\ldots,x_{T+m}$.

(c) Suppose $\theta_1 = -\lambda_1$. Show that this model is nothing but a representation of an i.i.d. series. What do you think will happen if you try to fit the ARMA(1,1) model, by least squares, to an i.i.d. series?

5. Consider a strictly stationary adapted process $\{x_t, \mathcal{F}_t\}$ where

$$E(x_t \mid \mathcal{F}_{t-1}) = \lambda x_{t-1} \text{ a.s.} \quad (1)$$

$$E((x_t - \lambda x_{t-1})^2 \mid \mathcal{F}_{t-1}) = \sigma^2 \text{ a.s.}, \quad (2)$$

λ and σ^2 are constant parameters and $|\lambda|<1,\ 0<\sigma^2<\infty$.

a) Derive the unconditional mean and unconditional variance of this process.

b) Show that ordinary least squares regression of x_t onto x_{t-1} yields a consistent estimator of λ.

c) Is the estimator also asymptotically normal? Set out the main steps of an argument to show that this property holds, supplying the additional assumptions required, if any.

d) Suppose that, instead of (1) and (2), you were only able to assume that the process was ergodic with finite variance, and

$$E(x_t x_{t-1}) = \lambda E(x_t^2).$$

Which, if any, of the results in parts (b) and (c) would still hold?

6. (a) What are the defining characteristics of Brownian motion? What other properties can be shown to follow from the definition?

(b) Explain the main steps in an argument to show that the process

$$X_T(r) = \frac{1}{\omega\sqrt{T}} \sum_{t=1}^{[Tr]} x_t, \quad 0 \le r \le 1$$

converges weakly to Brownian motion on the interval [0,1], where $\{x_t\}$ is a weakly dependent process and $0 < \omega < \infty$ where

$$\omega = \lim_{T\to\infty} T^{-1} E\left(\sum_{t=1}^{T} x_t\right)^2 .$$

Mention a set of conditions on $\{x_t\}$ that are sufficient for this result.

7.

(a) Explain the distinction between long memory and fractional integration.

(b) What is the spectral density function of the ARFIMA(*p*,*d*,*q*) model

$$\phi(L)(1-L)^d x_t = \theta(L)u_t,\ u_t \sim iid(0,\sigma^2)\ ?$$

Explain how the parameter *d* in this model can be estimated by log-periodogram regression.

(c) Discuss the relative merits of the log-periodogram regression estimator and the maximum likelihood estimator of the model in part (b).