Central European University

Advanced Time Series Analysis

Exercise 3: Martingales

1. Are the following processes $\{x_t,\ t > 0\}$ martingales, and why/why not? In each case take the relevant sequence of σ-fields to be $\mathcal{F}_t = \sigma(\varepsilon_t, \varepsilon_{t-1}, \varepsilon_{t-2}, \dots)$ for $-\infty < t < \infty$, where the driving process is $\varepsilon_t \sim NI(0, \sigma^2)$, and assume $Ex_0^2 < \infty$ where required.

   (a) $x_t = x_{t-1} + \varepsilon_t$

   (b) $x_t = x_{t-1}(1 + \varepsilon_t)$

   (c) $x_t = 0.5x_{t-1} + \varepsilon_t$

   (d) $x_t = x_{t-1} + \varepsilon_{t+1}$

   (e) $x_t = E(Z|\mathcal{F}_t)$ where Z is a random variable with $E|Z| < \infty$

   (f) $x_t = E(x_{t-1} + \varepsilon_{t+1}|\mathcal{F}_t)$.

2. Are the following processes martingale differences, and why/why not? If your answer is no, are they serially uncorrelated? (Same assumptions as in Qu. 1.)

   (a) $x_t = \varepsilon_{t-1}\varepsilon_t$

   (b) $x_t = \varepsilon_{t-1}^2\varepsilon_t$

   (c) $x_t = \varepsilon_{t-1}\varepsilon_t^2$

   (d) $x_t = \varepsilon_t^2 - \sigma^2$

   (e) $x_t = x_{t-1}\varepsilon_t, \quad t > 0$ where $x_0 = A \neq 0$, and $\sigma^2 = 1$.
   [Note: if $z \sim NI(0,1)$ then $E|z| = \sqrt{2/\pi} \approx 0.798$.]

3. Are these vector martingale differences? (Same assumptions).

   (a) $(\varepsilon_t,\ \varepsilon_t\varepsilon_{t-1})'$

   (b) $(\varepsilon_{t-1},\ \varepsilon_t\varepsilon_{t-1})'$

   (c) $(\varepsilon_t^3,\ \varepsilon_t\varepsilon_{t-1}^2)'$