Central European University

Advanced Time Series Analysis

Exercise 4 -Time Series Regression

1. Consider the AR(1) process

$$x_t = \lambda x_{t-1} + \varepsilon_t, \quad t = 1,2,3,\ldots$$

where $\varepsilon_t \sim NID(0,\sigma_\varepsilon^2)$ and x_0 is an independent $N(\mu_0,\sigma_0^2)$ random variable. Let $\{\mathcal{F}_t, t = 0,1,2,\ldots\}$ denote "information on the process up to date t" such that the pairs $\{x_t,\mathcal{F}_t\}$ form an adapted sequence.

(a) What are $E(x_t|\mathcal{F}_{t-1})$ and $\mathrm{Var}(x_t|\mathcal{F}_{t-1})$?

(b) What are $E(x_t|\mathcal{F}_0)$ and $\mathrm{Var}(x_t|\mathcal{F}_0)$?

(c) What are $E(x_t)$ and $\mathrm{Var}(x_t)$?

(d) What happens to $E(x_t)$ and $\mathrm{Var}(x_t)$ as t gets larger, in the cases
i) $|\lambda|< 1$, ii) $\lambda = 1$, iii) $\lambda > 1$?

(e) Find conditions on the distribution of x_0 such that the process is stationary (i.e. x_t has the same distribution for every $t \geq 1$).

2. Consider the model

$$y_t = \beta x_t + u_t, \quad t = 1,2,3,\ldots$$

where $u_t \sim NID(0,\sigma_u^2)$, x_t is generated as in Question 1. Write $\sigma_{u\varepsilon} = E(u_t\varepsilon_t)$.

(a) What are $E(y_t|\mathcal{F}_{t-1})$ and $\mathrm{Var}(y_t|\mathcal{F}_{t-1})$? (Remember that $\{y_t,\mathcal{F}_t\}$ are also adapted, by assumption.)

(b) Suppose u_t and ε_t are independent. Show that in this case $\hat{\beta}$, the least squares estimator of β, is consistent and asymptotically normal.

(c) Now assume that $\sigma_{u\varepsilon} \neq 0$. Show that in this case, $\hat{\beta}$ is inconsistent.

(d) Obtain the formula for $\tilde{\beta}$, the instrumental variables (IV) estimator in which x_{t-1} is used as the instrument.

(e) Is this estimator consistent? Is it asymptotically normal?

(f) Demonstrate why, if $\hat{\beta}$ were CAN, you would prefer it to $\tilde{\beta}$.

Answers to PS4.

1a) $E(x_t|\mathcal{F}_{t-1}) = \lambda x_{t-1}$, $\text{Var}(x_t|\mathcal{F}_{t-1}) = \sigma^2$

1b) $E(x_t|\mathcal{F}_0) = \lambda^t x_0$,

$$\begin{aligned}\text{Var}(x_t|\mathcal{F}_0) &= \sigma^2(1 + \lambda^2 + \cdots + \lambda^{2(t-1)}) \\ &= \sigma^2 \frac{1-\lambda^{2t}}{1-\lambda^2},\ \lambda \neq 1\end{aligned}$$

1c) $E(x_t) = \lambda^t \mu_0$,

$$\text{Var}(x_t) = \sigma^2 \frac{1-\lambda^{2t}}{1-\lambda^2} + \lambda^{2t}\sigma_0^2,$$

1d) (i)$E(x_t) \to 0$, $\text{Var}(x_t) \to \dfrac{\sigma^2}{1-\lambda^2}$. (ii) $E(x_t) = \mu_0$, all ,t but $\text{Var}(x_t) = t\sigma^2 + \sigma_0^2 \to \infty$, so mean not well-defined?
(iii) Both diverge.

1e)Set $\mu_0 = 0$, then $E(x_t) = 0$, all $t \geq 1$
Set $\sigma_0^2 = \dfrac{\sigma^2}{1-\lambda^2}$, then

$$\text{Var}(x_t) = \sigma^2 \frac{1-\lambda^{2t}}{1-\lambda^2} + \sigma^2 \frac{\lambda^{2t}}{1-\lambda^2} = \frac{\sigma^2}{1-\lambda^2},\ \text{all } t \geq 1$$

2. a) Note $y_t = \beta(\lambda x_{t-1} + \varepsilon_t) + u_t$

$$\begin{aligned}E(y_t|\mathcal{F}_{t-1}) &= \lambda\beta x_{t-1} \\ \text{Var}(y_t|\mathcal{F}_{t-1}) &= \beta^2\sigma_\varepsilon^2 + \sigma_u^2 + 2\beta\sigma_{\varepsilon u}\end{aligned}$$

2 b)

$$\hat{\beta} = \beta + \frac{T^{-1}\sum_{t=1}^T x_t u_t}{T^{-1}\sum_{t=1}^T x_t^2}$$

$$T^{-1}\sum x_t u_t = \lambda T^{-1}\sum x_{t-1}u_t + T^{-1}\sum \varepsilon_t u_t$$

Note, $E(x_{t-1}u_t|\mathcal{F}_{t-1}) = x_{t-1}E(u_t|\mathcal{F}_{t-1}) = 0$ a.s.
and $E|x_{t-1}u_t| \leq E(x_{t-1}^2 u_t^2) = \dfrac{\sigma_u^2\sigma_\varepsilon^2}{1-\lambda^2} < \infty$ hence $\{x_{t-1}u_t, \mathcal{F}_t\}$ is an m.d.
Similarly, $\{u_t\varepsilon_t\}$ is an i.i.d. sequence with means $\sigma_{u\varepsilon} = 0$ and variance $\sigma_u^2\sigma_\varepsilon^2$, hence an m.d. w.r.t. $\mathcal{F}_t$

m.d.s are uncorrelated, hence $T^{-1}\sum x_t u_t \xrightarrow{pr} 0$ by Chebyshev WLLN.

Also,

$$\begin{aligned} E(x_t u_t) &= \lambda^2 E(x_{t-1}^2 u_t^2) + E(\varepsilon_t^2 u_t^2) \\ &= \frac{\lambda^2 \sigma_u^2 \sigma_\varepsilon^2}{1-\lambda^2} + \sigma_u^2 \sigma_\varepsilon^2 \\ &= \sigma_u^2 \sigma_\varepsilon^2 \frac{\lambda^2 + 1 - \lambda^2}{1-\lambda^2} \\ &= \frac{\sigma_u^2 \sigma_\varepsilon^2}{1-\lambda^2}. \end{aligned}$$

Hence, $T^{-1/2}\sum x_t u_t \xrightarrow{d} N(0, \frac{\sigma_u^2 \sigma_\varepsilon^2}{1-\lambda^2})$ by CLT for m.d.s.

Also, consider

$$\begin{aligned} \frac{1}{T}\sum_{t=2}^T x_t^2 &= \lambda^2 \frac{1}{T}\sum_{t=2}^T x_{t-1}^2 + 2\frac{1}{T}\sum_{t=2}^T \lambda x_{t-1}\varepsilon_t + \frac{1}{T}\sum_{t=2}^T \varepsilon_t^2 \\ &= \frac{1}{1-\lambda^2}\left[2\frac{1}{T}\sum_{t=2}^T \lambda x_{t-1}\varepsilon_t + \frac{1}{T}\sum_{t=2}^T \varepsilon_t^2 + o_p(1) \right] \\ &\xrightarrow{pr} 0 + \frac{\sigma_\varepsilon^2}{1-\lambda^2} \end{aligned}$$

noting that

$$\frac{1}{T}\sum_{t=2}^T x_{t-1}^2 = \frac{1}{T}\sum_{t=2}^T x_t^2 + \frac{x_1^2 - x_T^2}{T}.$$

Hence, we get consistency (Slutsky) and asy-normality.

$$\sqrt{T}(\tilde{\beta} - \beta) \xrightarrow{d} N\left(0, \frac{\sigma_u^2 \frac{\sigma_\varepsilon^2}{1-\lambda^2}}{\left(\frac{\sigma_\varepsilon^2}{1-\lambda^2}\right)^2} \right)$$

where

$$\frac{\sigma_u^2 \frac{\sigma_\varepsilon^2}{1-\lambda^2}}{\left(\frac{\sigma_\varepsilon^2}{1-\lambda^2}\right)^2} = \frac{\sigma_u^2}{\sigma_\varepsilon^2}(1-\lambda^2)$$

3.Inconsistency:

$$\text{plim}\,\hat{\beta} - \beta = \frac{\sigma_{u\varepsilon}}{\sigma_\varepsilon^2/(1-\lambda^2)} \neq 0.$$

IV estimator:

$$\tilde{\beta} = \frac{\sum_{t=2}^T y_t x_{t-1}}{\sum_{t=2}^T x_t x_{t-1}} = \beta + \frac{T^{-1}\sum_{t=2}^T u_t x_{t-1}}{T^{-1}\sum_{t=2}^T x_t x_{t-1}}$$

$$E(u_t x_{t-1}|\mathcal{F}_{t-1}) = E(u_t|\mathcal{F}_{t-1})x_{t-1} = 0 \text{ a.s.}$$

(an m.d.) and

$$\begin{aligned} E(u_t^2 x_{t-1}^2) &= E((u_t^2|\mathcal{F}_{t-1})|x_{t-1}^2) \\ &= \sigma_u^2 \frac{\sigma_\varepsilon^2}{1-\lambda^2} \end{aligned}$$

$$\begin{aligned} T^{-1}\sum_{t=2}^T x_t x_{t-1} &= \lambda T^{-1}\sum_{t=2}^T x_{t-1}^2 + T^{-1}\sum_{t=2}^T \varepsilon_t x_{t-1} \\ &\overset{pr}{\to} \frac{\lambda\sigma_\varepsilon^2}{1-\lambda^2} + 0 \end{aligned}$$

Hence,

$$\sqrt{T}(\tilde{\beta} - \beta) \overset{d}{\to} N\left(0, \frac{\sigma_u^2 \frac{\sigma_\varepsilon^2}{1-\lambda^2}}{\left(\lambda \frac{\sigma_\varepsilon^2}{1-\lambda^2}\right)^2}\right)$$

where

$$\frac{\sigma_u^2 \frac{\sigma_\varepsilon^2}{1-\lambda^2}}{\left(\lambda \frac{\sigma_\varepsilon^2}{1-\lambda^2}\right)^2} = \frac{\sigma_u^2}{\sigma_\varepsilon^2}\frac{1-\lambda^2}{\lambda^2} > \frac{\sigma_u^2}{\sigma_\varepsilon^2}(1-\lambda^2)$$

so prefer OLS if consistent.