Central European University

Advanced Time Series Analysis

Exercise 6 - AR and ARMA Models

1. Suppose that a time series process $\{x_t, -\infty < t < \infty\}$ is observed to be strictly stationary and ergodic, with mean of zero and finite variance σ_x^2 , but its generation process is otherwise unknown. It is decided, for forecasting purposes, to fit a first-order autoregression to a realization of length T, as follows:

$$\hat{\lambda} = \frac{\sum_{t=2}^{T} x_t x_{t-1}}{\sum_{t=2}^{T} x_{t-1}^2} \tag{1}$$

   a) Show that $\hat{\lambda}$ is consistent for a parameter λ , and define λ.

   b) Is $\hat{\lambda}$ asymptotically normally distributed? Explain your answer.

   c) Consider the forecast

$$\hat{x}_{T+1} = \hat{\lambda} x_T$$

   where $\hat{\lambda}$ is given by (1). Show that the forecast error variance is approximately $\sigma_x^2(1 - \lambda^2)$ when T is large.

   d) Now suppose you have the additional information that $x_t - \lambda x_{t-1}$ is an identically and independently distributed shock sequence. Would your answers to (a) (b) and (c) be different in this case, and if so, how?

2. Consider the ARMA(1,1) process

$$x_t = \lambda_1 x_{t-1} + \varepsilon_t + \theta_1 \varepsilon_{t-1}, \quad \varepsilon_t \sim \mathrm{NI}(0, \sigma^2).$$

   a) Calculate the variance and first and second order autocovariances of this process. What can you say about the higher order autocovariances?

   b) Compare your formulae to those for the case where θ_1 is replaced by $1/\theta_1$ and σ^2 by $\omega^2 = \sigma^2\theta_1^2$.

   c) What do your results in (b) imply about estimation of the model?

   d) Suppose that $\lambda_1 = -\theta_1$. Comment on any pitfalls arising in the estimation of the model in this case.

The following experiments with TSM may be illuminate your answers to Question 2.

Start the program with the settings file TSAEx6.tsm and perform the simulation experiments set up there. The experimental models are:

1. ARMA(1,1)_0,2,.5 $\lambda_1 = 0, \theta_1 = 2, \sigma = 0.5.$
2. ARMA(1,1)_0,0,1 $\lambda_1 = 0, \theta_1 = 0, \sigma = 1.$
3. ARMA(1,1)_.5,0,1 $\lambda_1 = 0.5, \theta_1 = 0, \sigma = 1.$

The intercept has been omitted to save computing time, so there are three parameters in each model including the shock variance. Note that the variance parameter is entered in standard deviation form by default, so in the first model the variance is 0.25. The estimation method is Gaussian conditional maximum likelihood.

Each of the experiments has already been run with 1000 replications, with samples of size 200. The results of the last experiment run with it are stored with each model. Load a model in the Model Manager, and you can view the results in either tabular form (Setup /

Monte Carlo Experiments / <<Results >>) or graphical form (Graphics / Monte Carlo Distributions) If your computer is a bit slow, you can view the stored results instead of running the experiments again.

Question for discussion: To evaluate the likelihood function at the current parameter values (instead of optimizing it) press the  button. Try this with Model 1, at the stored initial values, and then after setting $\theta_1 = 0.5$, $\sigma = 1$. (Edit the parameter values in Values / Equation.) Is the result what you expect?

Answers:

1a) If x_t is stationary so are x_t^2 and $x_t x_{t-1}$. Hence by Ergodic Theorem

$$T^{-1}\sum x_t x_{t-1} \to_{as} E(x_t x_{t-1})$$
$$T^{-1}\sum x_{t-1}^2 \to_{as} E(x_t^2)$$

. by Slutsky Thm, $\hat{\lambda} \to_{pr} \gamma_1/\gamma_0 = \rho_1$.

b) No grounds to think this without further information.

c) Let $f_{T+!} = x_{T+1} - \hat{x}_{T+1} = x_{T+1} - \hat{\lambda} x_T$.

$$\begin{aligned} E(f_{T+1}^2) &= E(x_{T+1} - \hat{x}_{T+1})^2 \\ &\approx E(x_{T+1}^2) + \lambda^2 E(x_T^2) - 2\lambda E(x_T x_{T+1}) \\ &= \gamma_0 + \lambda^2\gamma_0 - 2\lambda\gamma_1 \\ &= \sigma_x^2(1 + \lambda^2 - 2\lambda^2) = \sigma_x^2(1 - \lambda^2) \end{aligned}$$

d)

$$\hat{\lambda} = \frac{\sum x_{t-1}x_t}{\sum x_{t-1}^2} = \lambda + \frac{\sum x_{t-1}u_t}{\sum x_{t-1}^2}$$

Consistency proof goes just as before! but note that λ is now defined by the model. Note that $Var(u_t) = \sigma_x^2(1-\lambda^2)$ same as 1-step forecast error variance.

Asymptotic normality:

$$E(x_{t-1}u_t|\mathcal{F}_{t-1}) = x_{t-1}E(u_t|\mathcal{F}_{t-1}) = 0 \text{ a.s.}$$

$$E(x_{t-1}^2 u_t^2) = E(x_{t-1}^2)E(u_t^2) = \sigma_x^2.\sigma_x^2(1-\lambda^2) < \infty$$

Hence $x_{t-1}u_t$ is integrable. Hence, it a stationary ergodic m.d. and

$$T^{-1/2}\sum x_{t-1}u_t \to_d N(0, \sigma_x^4(1-\lambda^2))$$

and $T^{-1/2}(\hat{\lambda} - \lambda) \to_d N(0,(1-\lambda^2))$ by Cramer thm.

2.

$$\begin{aligned}\gamma_0 &= E(x_t^2) = E(\lambda_1 x_{t-1} + \varepsilon_t + \theta_1\varepsilon_{t-1})^2 \\ &= E(\lambda_1^2 x_{t-1}^2 + \varepsilon_t^2 + \theta_1^2\varepsilon_{t-1}^2 + 2\lambda_1 x_{t-1}\varepsilon_t + 2\theta_1\lambda_1 x_{t-1}\varepsilon_{t-1} + 2\theta_1\varepsilon_{t-1\varepsilon_t}) \\ &= \lambda_1^2\sigma_x^2 + \sigma^2 + \theta_1^2\sigma^2 + 0 + 2\lambda_1\theta_1\sigma^2 + 0) \\ &= \sigma^2\frac{1+\theta_1^2+2\lambda_1\theta_1}{1-\lambda_1^2}\end{aligned}$$

$$\begin{aligned}\gamma_1 &= E(x_{t-1}(\lambda_1 x_{t-1} + \varepsilon_t + \theta_1\varepsilon_{t-1})) \\ &= \lambda_1\gamma_0 + 0 + \theta_1\sigma^2 \\ &= \sigma^2\left[\frac{\lambda_1(1+\theta_1^2+2\lambda_1\theta_1)}{1-\lambda_1^2} + \theta_1\right] = \sigma^2\left[\frac{\lambda_1 + \theta_1^2\lambda_1 + 2\lambda_1^2\theta_1 + \theta_1 - \theta_1\lambda_1^2}{1-\lambda_1^2}\right] \\ &= \sigma^2\left[\frac{\lambda_1 + 2\lambda_1^2\theta_1 + \theta_1}{1-\lambda_1^2}\right]\end{aligned}$$

$$\begin{aligned}\gamma_2 &= E(x_{t-2}(\lambda_1 x_{t-1} + \varepsilon_t + \theta_1\varepsilon_{t-1})) \\ &= \lambda_1\gamma_1,\end{aligned}$$

$\gamma_3 = \lambda_1\gamma_2$ etc.

b)

$$\begin{aligned}\gamma_0^* &= \sigma^2\theta_1^2\left(\frac{1+1/\theta_1^2+2\lambda_1/\theta_1}{1-\lambda_1^2}\right) = \gamma_0 \\ \gamma_1^* &= \sigma^2\theta_1^2\left(\frac{\lambda_1(1+1/\theta_1^2)+1/\theta_1}{1-\lambda_1^2}\right) \\ &= \sigma^2\left(\frac{\lambda_1(\theta_1^2+1)+\theta_1}{1-\lambda_1^2}\right) = \gamma_1\end{aligned}$$

Identical autocovariances, hence *same model* in the Gaussian case.

c) *Exact* likelihood must have equal maxima at points $(\lambda_1, \theta_1, \sigma^2)$ and $(\lambda_1, 1/\theta_1, \omega^2)$

d)

$$x_t - \lambda_1 x_{t-1} = \varepsilon_t - \lambda_1\varepsilon_{t-1}$$

or

$$x_t - \varepsilon_t = \lambda_1(x_{t-1} - \varepsilon_{t-1}) = \lambda_1(x_{t-2} - \varepsilon_{t-2}) = \cdots = 0$$

and

$$\gamma_0 = \sigma^2 \frac{1 + \lambda_1^2 - 2\lambda_1^2}{1 - \lambda_1^2} = \sigma^2$$

$$\gamma_1 = \sigma^2 \frac{\lambda_1(1 + \lambda_1^2) - \lambda_1(1 + \lambda_1^2)}{1 - \lambda_1^2} = 0$$

etc.