Central European University

Advanced Time Series Analysis

Exercise 7 - Unit Roots and Long Memory

1. a) What are the important properties of Brownian motion (BM)? Explain why these properties are the natural consequence of the fact that BM is the limit in distribution as $T \to \infty$ of the normalized partial sum process

$$X_n(r) = \frac{1}{\sigma\sqrt{T}} \sum_{j=1}^{[Tr]} u_j$$

where $u_j \sim \mathrm{NI}(0,\sigma^2)$.

b) Why do we expect the same properties to hold even if $\{u_j\}$ is not an independent Gaussian process, but is (say) a martingale difference?

c) If B denotes a BM, show that $\int_0^1 B(r)dr \sim N(0, \frac{1}{3})$.

2. Consider the long memory process with fixed starting date,

$$h_t = (1-L)^{-d} z_t^*$$

where $d > 0$,

$$z_t^* = \begin{cases} z_t & t \geq 1 \\ 0 & \text{otherwise} \end{cases}$$

and $z_t \sim \mathrm{iid}(\mu,\sigma^2)$.

a) Is this model linear? What are the properties of the lag coefficients?

b) Suppose $\mu = 0$. How does the variance of h_t for $t = 1,2,3,\dots$ depend on t?

c) Suppose $\mu \neq 0$. What can be said about the mean of h_t? Contrast this case with the case where $\mu = 0$ and

$$x_t = \alpha + (1-L)^{-d} z_t^*.$$

d) Consider the case $z_t = u_t^2$ where u_t is a disturbance, so that h_{t-1} represents a model of the conditional variance of u_t. Is this model plausible? Explain.

Answers:

1 a)

- continuous with probability 1. ($P(B \in C) = 1$

$B(0) = 0.$

1. Increments $B(r) - B(s)$ for $r > s$ are

- Totally independent when non-overlapping;
  $E(B(r+\delta+\eta) - B(r+\delta))(B(r+\delta) - B(r)) = 0$ for all $0 \le r < 1$, $0 < \delta \le 1-r$ and $0 < \eta \le 1-r-\delta$.
- Gaussian with mean 0 and variance $r - s$;

$$X_T(r+\delta) - X_T(r) = \frac{1}{\sigma\sqrt{T}} \sum_{j=[Tr\}+1}^{[T(r+\delta)} u_j$$

and

$$E(X_T(r+\delta) - X_T(r)) = \frac{1}{\sigma^2 T}\sigma^2[[T(r+\delta) - Tr]] = O(\delta)$$

The variance of an increment of width δ tends to 0 as $\delta \to 0$. Hence

$$Lim_{T\to\infty} P(|X_T(r+\delta) - X_T(r)|) = O(\delta)$$

by Chebyshev inequality, $P(|X|> \varepsilon) \le E(X^2)/\varepsilon$.

Gaussianity is by the fact that sums of independent Gaussians are Gaussian with summation rule for variances..

Independence of increments is by independence of the u_j.

a.s. continuity follows since Gaussian r.v.s are finite a.s.

b) Application of FCLT. The CLT means that increments are Gaussian in the limit. If u_t is m.d., the increments are uncorrelated:

$$E\left(\sum_{j=[Tr\}+1}^{[T(r+\delta)} u_j \middle| F_{[Tr]}\right) = 0$$

but for Gaussian r.v.s, uncorrelatredness implies independence!

c)

$$\int_0^1 X_n(r)dr = \sum_{t=0}^{T-1} \int_{t/T}^{t/T+1/T} \left(\frac{1}{\sigma\sqrt{T}} \sum_{j=1}^{[Tr]} u_j\right) dr$$

$$= \frac{1}{T} \sum_{t=1}^{T} \left(\frac{1}{\sigma\sqrt{T}} \sum_{j=1}^{t} u_j\right)$$

$$= \frac{1}{\sigma\sqrt{T}} \sum_{t=1}^{T} \frac{T-t+1}{T} u_t$$

Note, nonstationary (heteroscedastic) increments. However, it's sufficient for MDCLT if

$$T^{-1}\sum q_t^2 \to_{pr} \frac{\sigma^2}{3}.$$

We find, putting $s = T - t + 1$,

$$E\left(\frac{1}{\sqrt{T}}\sum_{t=1}^{T}\frac{T-t+1}{T}u_t\right)^2 = \frac{\sigma^2}{T^3}\sum_{t=1}^{T}(T-t+1)^2$$

$$= \frac{\sigma^2}{T^3}\sum_{s=1}^{T}s^2 = \sigma^2\frac{T(T+1)(2T+1)}{6T^3} \to \frac{\sigma^2}{3}$$

2. a)

$$h_t = \sum_{j=0}^{t-1} b_j z_{t-j}$$

where

$$b_j = \frac{\Gamma(d+j)}{\Gamma(d)\Gamma(1+j)} = O(j^{d-1})$$

Put $\mu = 0$:

$$E(h_t^2) = E\left(\sum_{j=0}^{t-1} b_j z_{t-j}\right)^2 = \sigma^2\sum_{j=0}^{t-1} b_j^2.$$

If $|d| < 1/2$ then since $2(d-1) < -1$,

$$E(h_t^2) \to \sigma^2\sum_{j=0}^{\infty} b_j^2 < \infty.$$

c)

$$E(h_t) = \mu\sum_{j=0}^{t-1} b_j = O(t^d)$$

Note summation rule:

$$\sum_{j=0}^{t-1} j^\alpha = \begin{cases} O(t^{\alpha+1}), \ \alpha > -1 \\ O(\log t), \ \alpha = -1 \\ = O(1), \ \alpha < -1 \end{cases}$$

In second case, $E(x_t) = \alpha$, all t.

d) Implies $h_t \to \infty$ as $t \to \infty$. Not plausible!