3.2 The Extension Theorem

You may wonder why, in the definition of a measurable space, $\mathcal{F}$ could not simply be the set of *all* subsets; the power set of Ω. The problem is to find a consistent method of assigning a measure to every set. This is straightforward when the space has a finite number of elements, but not in an infinite space where there is no way, even conceptually, to assign a specific measure to each set. It is necessary to specify a *rule* which generates a measure for any designated set. The problem of measurability is basically the problem of going beyond constructive methods without running into inconsistencies. We now show how this problem can be solved for σ-fields. These are a sufficiently general class of sets to cope with most situations arising in probability.

One must begin by assigning a measure, to be denoted μ_0, to the members of some basic collection $\mathcal{C}$ for which this can feasibly be done. For example, to construct Lebesgue measure we started by assigning to each interval $(a,b]$ the measure $b - a$. We then reason from the properties of μ_0 to extend it from this basic collection to all the sets of interest. $\mathcal{C}$ must be rich enough to allow μ_0 to be uniquely defined by it. A collection $\mathcal{C} \subseteq \mathcal{F}$ is called a *determining class* for $(\Omega,\mathcal{F})$ if, whenever μ and ν are measures on $\mathcal{F}$, $\mu(A) = \nu(A)$ for all $A \in \mathcal{C}$ implies that $\mu = \nu$.

Given $\mathcal{C}$, we must also know how to assign μ_0-values to any sets derived from $\mathcal{C}$ by operations such as union, intersection, complementation, and difference. For disjoint sets A and B we have $\mu_0(A \cup B) = \mu_0(A) + \mu_0(B)$ by finite additivity, and when $B \subseteq A$, $\mu_0(A - B) = \mu_0(A) - \mu_0(B)$. We also need to be able to determine $\mu_0(A \cap B)$, which will require specific knowledge of the relationship between the sets. When such assignments are possible for any pair of sets whose measures are themselves known, the measure is thereby extended to a wider class of sets, to be denoted $\mathcal{S}$. Often $\mathcal{S}$ and $\mathcal{C}$ are the same collection, but in any event $\mathcal{S}$ is closed under various finite set operations, and must at least be a semi-ring. In the applications $\mathcal{S}$ is typically either a field (algebra) or a semi-algebra. Example **1.18** is a good case to keep in mind.

However, $\mathcal{S}$ cannot be a σ-field since at most a finite number of operations are permitted to determine $\mu_0(A)$ for any $A \in \mathcal{S}$. At this point we might pose the opposite question to the one we started with, and ask why $\mathcal{S}$ might not be a rich enough collection for our needs. In fact, sets of interest frequently arise which $\mathcal{S}$ cannot contain. **3.15** below illustrates the necessity of being able to go to the limit, and consider sets that are expressible only as countably infinite unions or intersections of $\mathcal{C}$-sets. Extending to the sets $\mathcal{F} = \sigma(\mathcal{S})$ proves indispensable. We have two results, establishing existence and uniqueness respectively.

3.8 Extension theorem (existence) Let $\mathcal{S}$ be a semi-ring, and let $\mu_0 \colon \mathcal{S} \mapsto \overline{\mathbb{R}}^+$ be a measure on $\mathcal{S}$. If $\mathcal{F} = \sigma(\mathcal{S})$, there exists a measure μ on $(\Omega,\mathcal{F})$, such that $\mu(E) = \mu_0(E)$ for each $E \in \mathcal{S}$. □

Although the proof of the theorem is rather lengthy and some of the details are fiddly, the basic idea is simple. Take a set $A \subseteq \Omega$ to which we wish to assign a