on which μ and μ' are finite measures agreeing on $\mathcal{S} \cap B_j$. The preceding argument showed that, for $A \in \mathcal{F}$, $\mu(B_j \cap A) = \mu'(B_j \cap A)$ only if μ and μ' are the same measure.

Consider the following recursion. By **3.3**(ii) we have

$$\mu(A \cap (B_1 \cup B_2)) = \mu(A \cap B_1) + \mu(A \cap B_2) - \mu(A \cap B_1 \cap B_2). \tag{3.31}$$

Letting $C_n = \bigcup_{j=1}^n B_j$ the same relation yields

$$\mu(A \cap C_n) = \mu(A \cap B_n) + \mu(A \cap C_{n-1}) - \mu(A \cap B_n \cap C_{n-1}). \tag{3.32}$$

The terms involving C_{n-1} on the right-hand side can be solved backwards to yield an expression for $\mu(A \cap C_n)$, as a sum of terms having the general form

$$\mu(A \cap B_{j_1} \cap B_{j_2} \cap B_{j_3} \cap \ldots\,) = \mu(D \cap B_j) < \infty \tag{3.33}$$

for *some* j, say $j = j_1$, in which case $D = A \cap B_{j_2} \cap B_{j_3} \cap \ldots \in \mathcal{F}$. Since we know that $\mu(D \cap B_j) = \mu'(D \cap B_j)$ for all $D \in \mathcal{F}$ by the preceding argument, it follows that in (3.32)

$$\mu(A \cap C_n) = \mu'(A \cap C_n). \tag{3.34}$$

This holds for any n. Since $C_n \to \Omega$ as $n \to \infty$, we obtain in the limit

$$\mu(A) = \mu'(A), \tag{3.35}$$

the two sides of the equality being either finite and equal, or both equal to $+\infty$. This completes the proof, since A is arbitrary. ■

3.14 Example Let $\mathcal{M}$ denote the subsets of $\mathbb{R}$ which are measurable according to (3.14) when μ^* is the outer measure defined on the half-open intervals, whose measures μ_0 are taken equal to their lengths. This defines Lebesgue measure m. These sets form a semi-ring by **1.18**, a countable collection of them covers $\mathbb{R}$, and the extension theorem shows that, given m is a σ-finite measure, $\mathcal{M}$ contains the Borel field on $\mathbb{R}$ (see **1.21**), so $(\mathbb{R},\mathcal{B},m)$ is a measure space. It can be shown (we won't) that all the Lebesgue-measurable sets *not* in $\mathcal{B}$ are subsets of $\mathcal{B}$-sets of measure 0. □

The following is a basic property of Lebesgue measure. Notice the need to deal with a countable intersection of intervals to determine so simple a thing as the measure of a point.

3.15 Theorem Any countable set from $\mathbb{R}$ has Lebesgue measure 0.

Proof The measure of a point $\{x\}$ is zero, since for $x \in \mathbb{R}$,

$$\{x\} = \bigcap_{n=1}^{\infty}(x - 1/n,\ x] \in \mathcal{B} \tag{3.36}$$

and $m(\{x\}) = \lim_{n\to\infty} 1/n = 0$. The result follows by **3.6**(ii). ■