For finite n at least, a separate theory is not needed because results can be obtained by recursion. If $(\Psi,\mathcal{H})$ is a third measurable space, then trivially,

$$\begin{aligned}\Omega\times\Xi\times\Psi &= \{(\omega,\xi,\psi)\colon \omega\in\Omega,\ \xi\in\Xi,\ \psi\in\Psi\} \\ &= \{((\omega,\xi),\psi)\colon (\omega,\xi)\in\Omega\times\Xi,\ \psi\in\Psi\} \\ &= (\Omega\times\Xi)\times\Psi. \end{aligned}\tag{3.49}$$

Either or both of $(\Omega,\mathcal{F})$ and $(\Xi,\mathcal{G})$ can be product spaces, and the last two theorems extend to product spaces of any finite dimension.

3.5 Measurable Transformations

Consider measurable spaces $(\Omega,\mathcal{F})$ and $(\Xi,\mathcal{G})$ in a different context, as domain and codomain of a mapping

$$T\colon \Omega \mapsto \Xi.$$

T is said to be *$\mathcal{F}/\mathcal{G}$-measurable* if $T^{-1}(B)\in\mathcal{F}$ for all $B\in\mathcal{G}$. The idea is that a measure μ defined on $(\Omega,\mathcal{F})$ can be mapped into $(\Xi,\mathcal{G})$, every set $B\in\mathcal{G}$ being assigned a measure $\nu(B)=\mu(T^{-1}(B))$. We have just encountered one example, the projection mapping, whose inverse defined in (3.44) takes each $\mathcal{F}$-set A into a measurable rectangle.

Corresponding to a measurable transformation there is always a transformed measure, in the following sense.

3.21 Theorem Let μ be a measure on $(\Omega,\mathcal{F})$ and $T\colon\Omega\mapsto\Xi$ a measurable transformation. Then μT^{-1} is a measure on $(\Xi,\mathcal{G})$, where

$$\mu T^{-1}(B) = \mu(T^{-1}(B)),\ \text{each } B\in\mathcal{G}. \tag{3.50}$$

Proof We check conditions **3.1**(a)-(c). Clearly $\mu T^{-1}(A)\ge 0$, all $A\in\mathcal{B}_T$. Since $T^{-1}(\Xi)=\Omega$ holds by definition, $T^{-1}(\varnothing)=\varnothing$ by **1.2**(iii) and so $\mu T^{-1}(\varnothing) = \mu(T^{-1}(\varnothing)) = \mu(\varnothing) = 0$. For countable additivity we must show

$$\mu T^{-1}\left(\bigcup_j B_j\right) = \sum_j \mu T^{-1}(B_j) \tag{3.51}$$

for a disjoint collection $B_1,B_2,\ldots\in\Xi$. Letting $B_j' = T^{-1}(B_j)$, **1.2** shows both that the B_j' are disjoint and that $T^{-1}\left(\bigcup_j B_j\right) = \bigcup_j B_j'$. Equation (3.51) therefore becomes

$$\mu\left(\bigcup_j B_j'\right) = \sum_j \mu(B_j') \tag{3.52}$$

for disjoint sets B_j', which holds because μ is a measure. ■

The main result on general transformations is the following.

3.22 Theorem Suppose $T^{-1}(B)\in\mathcal{F}$ for each $B\in\mathcal{D}$, where $\mathcal{D}$ is an arbitrary class of sets, and $\mathcal{G}=\sigma(\mathcal{D})$. Then the transformation T is $\mathcal{F}/\mathcal{G}$-measurable.