7
Probability Spaces

7.1 Probability Measures

A *random experiment* is an action or observation whose outcome is uncertain in advance of its occurrence. Tosses of a coin, spins of a roulette wheel, and observations of the price of a stock are familiar examples. A *probability space*, the triple $(\Omega,\mathcal{F},P)$, is to be thought of as a mathematical model of a random experiment. Ω is the *sample space*, the set of all the possible outcomes of the experiment, called the *random elements*, individually denoted ω. The collection $\mathcal{F}$ of *random events* is a σ-field of subsets of Ω, the event $A \in \mathcal{F}$ being said to have occurred if the outcome of the experiment is an element of A. A measure P is assigned to the elements of $\mathcal{F}$, $P(A)$ being the *probability* of A. Formally, we have the following.

7.1 Definition A probability measure (p.m.) on a measurable space $(\Omega,\mathcal{F})$ is a set function $P\colon \mathcal{F} \mapsto [0,1]$ satisfying the *axioms of probability*:

(a) $P(A) \geq 0$, for all $A \in \mathcal{F}$.

(b) $P(\Omega) = 1$.

(c) Countable additivity: for a disjoint collection $\{A_j \in \mathcal{F},\ j \in \mathbb{N}\}$,

$$P\left(\bigcup_j A_j\right) = \sum_j P(A_j). \ \square \tag{7.1}$$

Under the frequentist interpretation of probability, $P(A)$ is the limiting case of the proportion of a long run of repeated experiments in which the outcome is in A. Alternatively, probability may be viewed as a subjective notion with $P(A)$ said to represent an observer's degree of belief that A will occur. For present purposes, the interpretation given to the probabilities has no relevance. The theory stands or falls by its mathematical consistency alone, although it is then up to us to decide whether the results accord with our intuition and are useful in the analysis of real-world problems.

Additional properties of P follow from the axioms.

7.2 Theorem If A, B, and $\{A_j,\ j \in \mathbb{N}\}$ are arbitrary $\mathcal{F}$-sets, then

(i) $P(A) \leq 1$.

(ii) $P(A^c) = 1 - P(A)$.

(iii) $P(\varnothing) = 0$.

(iv) $A \subseteq B \ \Rightarrow\ P(A) \leq P(B)$ (monotonicity).

(v) $P(A \cup B) = P(A) + P(B) - P(A \cap B)$.

(vi) $P\left(\bigcup_j A_j\right) \leq \sum_j P(A_j)$ (countable subadditivity).