10.20 Theorem Let a function $G(\omega,\theta)$ satisfy the conditions of **9.31**. Then

$$E\left(\frac{dG}{d\theta}\bigg|_{\theta=\theta_0}\bigg|\,\mathcal{G}\right) = \frac{dE(G|\mathcal{G})}{d\theta}\bigg|_{\theta=\theta_0}, \text{ a.s.} \tag{10.49}$$

Proof Take a countable sequence $\{h_\nu, \nu \in \mathbb{N}\}$ with $h_\nu \to 0$ as $\nu \to \infty$. By linearity of the conditional expectation,

$$E\left(\frac{G(\theta_0+h_\nu) - G(\theta_0)}{h_\nu}\bigg|\,\mathcal{G}\right) = \frac{E\big(G(\theta_0+h_\nu)|\mathcal{G}\big) - E\big(G(\theta_0)|\mathcal{G}\big)}{h_\nu} \text{ a.s.} \tag{10.50}$$

If $C_\nu \in \mathcal{G}$ is the set on which the equality in (10.50) holds, with $P(C_\nu) = 1$, the two sequences agree in the limit on the set $\bigcap_\nu C_\nu$, and $P(\bigcap_\nu C_\nu) = 1$ by **3.6**. The left-hand side of (10.50) converges a.s. to the left-hand side of (10.49) by assumption, applying the conditional version of the dominated convergence theorem. Since whenever it exists the a.s. limit of the right-hand side of (10.50) is the right-hand side of (10.49) by definition, the theorem follows. ∎

10.5 Relationships between Subfields

$\mathcal{G}_1 \subseteq \mathcal{F}$ and $\mathcal{G}_2 \subseteq \mathcal{F}$ are *independent* subfields if, for every pair of events $G_1 \in \mathcal{G}_1$ and $G_2 \in \mathcal{G}_2$,

$$P(G_1 \cap G_2) = P(G_1)P(G_2). \tag{10.51}$$

Note that if Y is measurable on $\mathcal{G}_1$ it is also measurable on any collection containing $\mathcal{G}_1$, and on $\mathcal{F}$ in particular. Theorems **10.10** and **10.11** cover cases where Y as well as X is measurable on a subfield.

10.21 Theorem Random variables X and Y are independent iff $\sigma(X)$ and $\sigma(Y)$ are independent.

Proof Under the inverse mapping in (10.13), $G_1 \in \sigma(X)$ if and only if $B_1 = X(G_1) \in \mathcal{B}$ with a corresponding condition for $\sigma(Y)$. It follows that (10.51) holds for each $G_1 \in \sigma(X)$, $G_2 \in \sigma(Y)$ iff $P(X \in B_1, Y \in B_2) = P(X \in B_1)P(Y \in B_2)$ for each $B_1 = X(G_1)$, $B_2 = Y(G_2)$. The 'only if' of the theorem then follows directly from the definition of $\sigma(X)$. The 'if' follows, given (8.38), from the fact that every $B_i \in \mathcal{B}$ has an inverse image in any subfield on which a r.v. is measurable. ∎

The 'only if' in the first line of this proof is essential. Independence of the subfields always implies independence of X and Y, but the converse holds only for the infimal cases, $\sigma(X)$ and $\sigma(Y)$.

10.22 Theorem Let Y be integrable and measurable on $\mathcal{G}_1$. Then $E(Y|\mathcal{G}) = E(Y)$ a.s. for all $\mathcal{G}$ independent of $\mathcal{G}_1$.

Proof Define the simple $\mathcal{G}_1$-measurable r.v.s $Y_{(n)} = \sum_{i=1}^n \gamma_i 1_{G_{1i}}$ on a partition $G_{11},\dots,G_{1n}$ of Ω where $G_{1i} \in \mathcal{G}_1$, each i, with $Y_{(n)} \uparrow Y$ as in **3.28**. Then