12.3 Theorem $\sigma(\mathscr{C}) = \mathcal{B}^\infty$, the Borel field of sets in $\mathbb{R}^\infty$ with the Tychonoff topology. □

Fig. 12.2

The condition of this result is something we can take for granted in the usual applications. Recalling that the Borel field of a space is the smallest σ-field containing the open sets, **12.3** is true by definition, since $\mathscr{C}$ is a sub-base for the product topology (see §6.5) and all the open sets of $\mathbb{R}^\infty$ are generated by unions and finite intersections of $\mathscr{C}$-sets. To avoid explicit topological considerations, the reader may like to think of **12.3** as providing the definition of $\mathcal{B}^\infty$.

One straightforward implication, since the coordinate projections are continuous mappings and hence measurable, is that, given a distribution on $(\mathbb{R}^\infty,\mathcal{B}^\infty)$, finite collections of sequence coordinates can always be treated as random vectors. But, while this is obviously a condition that will need to be satisfied, the real problem runs the other way. The only *practical* method we have of defining distributions for infinite sequences is to assign probabilities to finite collections of coordinates, after the manner of §8.4. The serious question is whether this can be done in a consistent manner, so that in particular, there exists a p.m. on $(\mathbb{R}^\infty,\mathcal{B}^\infty)$ that corresponds to a given set of the finite-dimensional distributions. The affirmative answer to this question is the famous Kolmogorov consistency theorem.

12.4 The Consistency Theorem

The goal is to construct a p.m. on $(\mathbb{R}^\infty,\mathcal{B}^\infty)$, and, following the approach of §3.2, the plausible first step in this direction is to assign probabilities to elements of $\mathscr{C}$. Let μ_k denote a p.m. on the space $(\mathbb{R}^k,\mathcal{B}^k)$, for k = 1,2,3,.... We will say that this family of measures satisfies the *consistency property* if

$$\mu_k(E) = \mu_m(E\times\mathbb{R}^{m-k}) \tag{12.7}$$