Proof Let $\mathcal{G}^t_{-\infty} = \sigma(\dots,Y_{t-1},Y_t)$, and $\mathcal{G}^\infty_{t+m} = \sigma(Y_{t+m},Y_{t+m+1},\dots)$. Since Y_t is measurable on any σ-field on which each of $X_t,X_{t-1},\dots,X_{t-\tau}$ are measurable, $\mathcal{G}^t_{-\infty} \subseteq \mathcal{F}^t_{-\infty}$ and $\mathcal{G}^\infty_{t+m} \subseteq \mathcal{F}^\infty_{t+m-\tau}$. Let $\alpha_{Y,m} = \sup_t \alpha(\mathcal{G}^t_{-\infty},\mathcal{G}^\infty_{t+m})$ and it follows that $\alpha_{Y,m} \le \alpha_{m-\tau}$ for $m \ge \tau$. With τ finite, $\alpha_{m-\tau} = O(m^{-\varphi})$ if $\alpha_m = O(m^{-\varphi})$ and the conclusion follows. The same argument follows word for word with 'ϕ' replacing 'α'. ∎

14.2 Mixing Inequalities

Strong and uniform mixing are restrictions on the complete joint distribution of the sequence, and to make practical use of the concepts we must know what they imply about particular measures of dependence. This section establishes a set of fundamental moment inequalities for mixing processes. The main results bound the m-step-ahead predictions, $E(X_{t+m}|\mathcal{F}^t_{-\infty})$. Mixing implies that, as we try to forecast the future path of a sequence from knowledge of its history to date, looking further and further forward, we will eventually be unable to improve on the predictor based solely on the distribution of the sequence as a whole, $E(X_{t+m})$. The r.v. $E(X_{t+m}|\mathcal{F}^t_{-\infty}) - E(X_{t+m})$ is tending to zero as m increases. We prove convergence of the L_p-norm.

14.2 Theorem (Ibragimov 1962) For $r \ge p \ge 1$ and with α_m defined in (14.1),

$$\|E(X_{t+m}|\mathcal{F}^t_{-\infty}) - E(X_{t+m})\|_p \le 2(2^{1/p}+1)\alpha_m^{1/p-1/r}\|X_{t+m}\|_r. \tag{14.7}$$

Proof To simplify notation, substitute X for X_{t+m}, $\mathcal{G}$ for $\mathcal{F}^t_{-\infty}$, $\mathcal{H}$ for $\mathcal{F}^\infty_{t+m}$, and α for α_m. It will be understood that X is an $\mathcal{H}$-measurable random variable where $\mathcal{G},\mathcal{H} \subseteq \mathcal{F}$. The proof is in two stages, first to establish the result for $|X| \le M_X < \infty$ a.s., and then to extend it to the case where X is L_r-bounded for finite r. Define the $\mathcal{G}$-measurable r.v.

$$\eta = \operatorname{sgn}(E(X|\mathcal{G}) - E(X)) = \begin{cases} 1, & E(X|\mathcal{G}) \ge E(X), \\ -1, & \text{otherwise.} \end{cases} \tag{14.8}$$

Using **10.8** and **10.10**,

$$\begin{aligned} E|E(X|\mathcal{G}) - E(X)| &= E\big[\eta(E(X|\mathcal{G}) - E(X))\big] \\ &= E\big[(E(\eta X|\mathcal{G}) - \eta E(X)\big] \\ &= \mathrm{Cov}(\eta,X) = |\mathrm{Cov}(\eta,X)|. \end{aligned} \tag{14.9}$$

Let Y be any $\mathcal{G}$-measurable r.v., such as η for example. Noting that $\xi = \operatorname{sgn}(E(Y|\mathcal{H}) - E(Y))$ is $\mathcal{H}$-measurable, similar arguments give

$$\begin{aligned} |\mathrm{Cov}(X,Y)| &= |E\big(X(E(Y|\mathcal{H}) - E(Y))\big)| \\ &\le E\big(|X||(E(Y|\mathcal{H}) - E(Y))|\big) \\ &\le M_X E|E(Y|\mathcal{H}) - E(Y)| \end{aligned}$$