This is appealing at an elementary level since it captures the notion of information available to an observer, in this case the sequence realization to date. But since, as we have seen, the conditioning information can extend more widely than the history of the sequence itself, this type of notation is relatively clumsy. Suppose we have a vector sequence $\{(X_t,Z_t)\}$, and X_t—though not necessarily Z_t—is a m.d. with respect to $\mathcal{F}_t = \sigma(X_t,Z_t,X_{t-1},Z_{t-1},...)$ in the sense of (15.6). This case is distinct from (15.12), and shows that that definition is inadequate, although (15.6) implies (15.12). More important, the representation of conditioning information is not unique, and we have seen (**10.3**(ii)) that any measurably isomorphic transformation of the conditioning variables contains the same information as the original variables. Indeed, the information need not even be represented by a variable, but is merely knowledge of the occurrence/non-occurrence of certain abstract events.

15.2 Extensions of the Martingale Concept

An adapted triangular array $\{\{X_{nt},\mathcal{F}_{nt}\}_{t=1}^{k_n}\}_{n=1}^{\infty}$, where $\{k_n\}_{n=1}^{\infty}$ is some increasing sequence of integers, for which

$$E|X_{nt}| < \infty, \tag{15.13}$$

$$E(X_{nt}|\mathcal{F}_{n,t-1}) = 0 \text{ a.s.} \tag{15.14}$$

for each $t = 1,...,k_n$ and $n \geq 1$, is called a *martingale difference array*. In many applications we would have just $k_n = n$. The double subscripting of the subfield $\mathcal{F}_{nt}$ may be superfluous if the information content of the array does not depend on n, with $\mathcal{F}_{nt} = \mathcal{F}_t$ for each n, but the additional generality given by the definition is harmless and could be useful. The sequence $\{S_n,\mathcal{F}_n\}_1^{\infty}$ where $S_n = \sum_{t=1}^{k_n}X_{nt}$ and $\mathcal{F}_n = \mathcal{F}_{n,k_n}$ is not a martingale, but the properties of martingales can be profitably used to analyse its behaviour. Consider the case $S_n = n^{-1/2}\sum_{t=1}^{n}X_t$ where $\{X_t,\mathcal{F}_t\}$ is a m.d. Such scaling by sample size may ensure that the distribution of S_n has a non-degenerate limit. S_n is not a martingale since

$$E(S_n|\mathcal{F}_{n-1}) = [(n-1)/n]^{1/2}S_{n-1}, \tag{15.15}$$

but each column of the m.d. array

$$\begin{bmatrix} X_1 & 2^{-1/2}X_1 & 3^{-1/2}X_1 & 4^{-1/2}X_1 & \cdots \\ & 2^{-1/2}X_2 & 3^{-1/2}X_2 & 4^{-1/2}X_2 & \cdots \\ & & 3^{-1/2}X_3 & 4^{-1/2}X_3 & \cdots \\ & & & 4^{-1/2}X_4 & \cdots \\ & & & & \ddots \end{bmatrix} \tag{15.16}$$

is a m.d. sequence, and S_n is the sum of column n. It is a term in a martingale sequence even though this is not the sequence $\{S_n\}$.

An adapted sequence $\{S_n,\mathcal{F}_n\}_{-\infty}^{\infty}$ of L_1-bounded variables satisfying