The technical argument in the final paragraph of this proof can be better appreciated after studying Chapter 18. It is neither possible nor necessary in this approach to assert that $E(W_{mt}|\mathcal{F}_{t-1}) \to 0$ a.s.

Note how taking conditional expectations of (16.12) yields

$$E(X_t|\mathcal{F}_{t-1}) = Z_t - Z_{t+1} \text{ a.s.} \tag{16.21}$$

It follows that W_t is almost surely equal to the centred r.v. $X_t - E(X_t|\mathcal{F}_{t-1})$.

16.7 Example Consider the linear process from **16.2**, with $\{U_t\}$ a stationary integrable sequence. Then X_t is stationary, and

$$E|X_1| \le E|U_1|\sum_{j=-\infty}^{\infty}|\theta_j| < \infty.$$

If the coefficients satisfy a stronger summability condition, i.e.

$$\sum_{m=1}^{\infty}\sum_{j=m}^{\infty}(|\theta_j| + |\theta_{-j}|) = \sum_{m=1}^{\infty}m|\theta_m| + \sum_{m=1}^{\infty}m|\theta_{-m}| < \infty, \tag{16.22}$$

then X_t is an L_1-mixingale of size -1. By a rearrangement of terms we obtain the decomposition of (16.12) with

$$W_t = \left(\sum_{j=-\infty}^{\infty}\theta_j\right)U_t \tag{16.23}$$

and

$$Z_t = \sum_{m=1}^{\infty}\left(\left(\sum_{j=m}^{\infty}\theta_j\right)U_{t-m} - \left(\sum_{j=m}^{\infty}\theta_{-j}\right)U_{t+m-1}\right), \tag{16.24}$$

where $E|Z_t| < \infty$ by (16.22). □

16.3 Maximal Inequalities

As with martingales, maximal inequalities are central to applications of the mixingale concept in limit theory. The basic idea of these results is to extend Doob's inequality (**15.15**) by exploiting the representation as a telescoping sum of martingale differences. McLeish's idea is to let m go to ∞ in (16.10), and accordingly write

$$S_n = \sum_{k=-\infty}^{\infty}Y_{nk}, \text{ a.s.} \tag{16.25}$$

16.8 Lemma Suppose $\{S_j\}_1^n$ has the representation in (16.25). Let $\{a_k\}_{-\infty}^{\infty}$ be a summable collection of non-negative real numbers, with $a_k = 0$ if $Y_{nk} = 0$ a.s., and $a_k > 0$ otherwise. For any $p > 1$,

$$E\left(\max_{1\le j\le n}|S_j|^p\right) \le \left(\frac{p}{p-1}\right)^p\left(\sum_{k=-\infty}^{\infty}a_k\right)^{p-1}\sum_{a_k>0}a_k^{1-p}E|Y_{nk}|^p. \tag{16.26}$$