of a class of convergence results they are rather restrictive. We can set $\mu = 0$ with no loss of generality, by simply considering the centred sequence $\{X_t - \mu_t\}_1^\infty$; centring is generally a good idea, because then it is no longer necessary for the time average of the means to converge in the manner specified. We can quite easily have $n^{-1}\sum_{t=1}^n \mu_t \longrightarrow \infty$ at the same time that $n^{-1}\sum_{t=1}^n (X_t(\omega) - \mu_t) \longrightarrow 0$. In such cases the law of large numbers requires a modified interpretation, since it ceases to make sense to speak of convergence of the sequence of sample means.

More general modes of convergence also exist. It is possible that $\bar{X}_n$ does not converge in the manner specified, even after centring, but that there exists a sequence of positive constants $\{a_n\}_1^\infty$ such that $a_n \uparrow \infty$ and $a_n^{-1}\sum_{t=1}^n X_t \longrightarrow 0$. Results below will subsume these possibilities, and others too, in a fully general array formulation of the problem. If $\{\{X_{nt}\}_{t=1}^{k_n}\}_{n=1}^\infty$ is a triangular stochastic array with $\{k_n\}_{n=1}^\infty$ an increasing integer sequence, we will discuss conditions for

$$S_n = \sum_{t=1}^{k_n} X_{nt} \xrightarrow{pr} 0. \tag{18.33}$$

A result in this form can be specialized to the familiar case with $X_{nt} = a_n^{-1}(X_t - \mu_t)$ and $a_n = k_n = n$, but there are important applications where the greater generality is essential.

We have already encountered two cases where the strong law of large numbers applies. According to **13.12**, $\bar{X}_n \xrightarrow{as} \mu = E(X_1)$ when $\{X_t\}$ is a stationary ergodic sequence and $E|X_1| < \infty$. We can illustrate the application of this type of result by an example in which the sequence is independent, which is sufficient for ergodicity.

18.17 Example Consider a sequence of independent Bernoulli variables X_t with $P(X_t = 1) = P(X_t = 0) = \frac{1}{2}$; that is, of coin tosses expressed in binary form (see **12.1**). The conditions of the ergodic theorem are clearly satisfied, and we can conclude that $n^{-1}\sum_{t=1}^n X_t \xrightarrow{as} E(X_t) = \frac{1}{2}$. This is called *Borel's normal number theorem*, a normal number being defined as one in which 0s and 1s occur in its binary expansion with equal frequency, in the limit. The normal number theorem therefore states that almost every point of the unit interval is a normal number; that is, the set of normal numbers has Lebesgue measure 1.

Any number with a terminating expansion is clearly non-normal and we know that all such numbers are rationals; however, rationals can be normal, as for example $\frac{1}{3}$, which has the binary expansion 0.01010101010101... This is a different result from the well-known zero measure of the rationals, and is much stronger, because the non-normal numbers include irrationals, and form an uncountable set. For example, any number with a binary expansion of the form $0.11b_1 11b_2 11b_3 11...$ where the b_i are arbitrary digits is non-normal; yet this set can be put into 1-1 correspondence with the expansions $0.b_1b_2b_3,...$, in other words, with the points of the whole interval. The set of non-normal numbers is equipotent with the reals, but it none the less has Lebesgue measure 0. □

A useful fact to remember is that the stationary ergodic propery is preserved under measurable transformations; that is, if $\{X_t\}$ is stationary and ergodic, so