19.10 Corollary If $\{X_{nt}\}$ is a zero-mean stochastic array with $E(X_{nt}X_{ns}) = 0$ for $t \neq s$, and

(a) $\{X_{nt}/c_{nt}\}$ is uniformly L_2-bounded, and

(b) $\lim_{n\to\infty} \sum_{t=1}^{k_n} c_{nt}^2 = 0$,

then $\sum_{t=1}^{k_n} X_{nt} \xrightarrow{L_2} 0$. □

19.4 A Mixingale Weak Law

To generalize the last results from martingale differences to mixingales is not too difficult. The basic tool is the 'telescoping series' argument developed in §16.2. The array element X_{nt} can be decomposed into a finite sum of martingale differences, to which **19.7** can be applied, and two residual components which can be treated as negligible. The following result, from Davidson (1993a), is an extension to the heterogeneous case of a theorem due to Andrews (1988).

19.11 Theorem Let the array $\{X_{nt},\mathcal{F}_{nt}\}_{-\infty}^{\infty}$ be a L_1-mixingale with respect to a constant array $\{c_{nt}\}$. If

(a) $\{X_{nt}/c_{nt}\}$ is uniformly integrable,

(b) $\limsup_{n\to\infty} \sum_{t=1}^{k_n} c_{nt} < \infty$, and

(c) $\lim_{n\to\infty} \sum_{t=1}^{k_n} c_{nt}^2 = 0$,

where k_n is an increasing integer-valued function of n and $k_n \uparrow \infty$, then $\sum_{t=1}^{k_n} X_{nt} \xrightarrow{L_1} 0$. □

There is no restriction on the size here. It suffices simply for the mixingale coefficients to tend to zero. The remarks following **19.7** apply here in just the same way. In particular, if X_t is a L_1-mixingale sequence and $\{X_t/b_t\}$ is uniformly integrable for positive constants $\{b_t\}$, the theorem holds for $X_{nt} = X_t/a_n$ and $c_{nt} = b_t/a_n$ where $a_n = \sum_{t=1}^{n} b_t$. Theorems **14.2** and **14.4** give us the corresponding results for mixing sequences, and **17.5** and **17.6** for NED processes. It is sufficient for, say, X_{nt} to be L_r-bounded for $r > 1$, and L_p-NED, for $p \geq 1$, on a α-mixing process. Again, no size restrictions need to be specified. Uniform integrability of $\{X_{nt}/c_{nt}\}$ will obtain in those cases where $\|X_{nt}\|_r$ is finite for $r > 1$ and each t, and the NED constants likewise satisfy $d_{nt} \gg \|X_{nt}\|_r$.

A simple lemma is required for the proof:

19.12 Lemma If the array $\{X_{nt}/c_{nt}\}$ is uniformly integrable, so is the array $\{E_{t-j}X_{nt}/c_{nt}\}$ for $j > 0$.

Proof By the necessity part of **12.9**, for any $\varepsilon > 0$ $\exists$ $\delta > 0$ such that