20
The Strong Law of Large Numbers

20.1 Technical Tricks for Proving LLNs

In this chapter we explore the strong law under a range of different assumptions, from independent sequences to near-epoch dependent functions of mixing processes. Many of the proofs are based on one or more of a collection of ingenious technical lemmas, and we begin by studying these results. The reader has the option of skipping ahead to §20.2, and referring back as necessary, but there is something to be said for forming an impression of the method of attack at the outset. These theorems are found in several different versions in the literature, usually in a form adapted to the particular problem in hand. Here we will take note of the minimal conditions needed to make each trick work.

We start with the basic convergence result that shows why maximal inequalities (for example, **15.14**, **15.15**, **16.9**, and **16.11**) are important.

20.1 Convergence lemma Let $\{X_t\}_1^\infty$ be a stochastic sequence on a probability space $(\Omega,\mathcal{F},P)$, and let $S_n = \sum_{t=1}^n X_t$ and $S_0 = 0$. For $\omega \in \Omega$, let

$$M(\omega) = \inf_m \left(\sup_{j>m} |S_j(\omega) - S_m(\omega)|\right). \tag{20.1}$$

If $P(M > \varepsilon) = 0$ for all $\varepsilon > 0$, then $S_n \xrightarrow{as} S$.

Proof By the Cauchy criterion for convergence**,** the realization $\{S_n(\omega)\}$ converges if we can find an m such that $|S_j - S_m| \le \varepsilon$ for all $j > m$, for all $\varepsilon > 0$; in other words, it converges if $M(\omega) \le \varepsilon$ for all $\varepsilon > 0$. ∎

This result is usually applied in the following way.

20.2 Corollary Let $\{c_t\}_1^\infty$ be a sequence of constants, and suppose there exists $p > 0$ such that, for every $m \ge 0$ and $n > m$, and every $\varepsilon > 0$,

$$P\left(\max_{m<j\le n} |S_j - S_m| > \varepsilon\right) \le \frac{K}{\varepsilon^p} \sum_{t=m+1}^n c_t^p, \tag{20.2}$$

where K is a finite constant. If $\sum_{t=1}^\infty c_t^p < \infty$, then $S_n \xrightarrow{as} S$.

Proof Since $\{c_t^p\}$ is summable it follows by **2.25** that $\lim_{m\to\infty}\sum_{t=m+1}^\infty c_t^p = 0$. Let M be the r.v. in (20.1). By definition, $M \le \sup_{j>m}|S_j - S_m|$ for any $m > 0$, and hence

$$P(M > \varepsilon) \le \lim_{m\to\infty} P\left(\sup_{j>m}|S_j - S_m| > \varepsilon\right)$$