the weight sequence is also allowed to be stochastic. The style of this result is appropriate to the class of martingale limit theorems we shall examine in §20.4, in which we establish almost-sure equivalence between sets on which certain conditions obtain and on which sequences converge.

20.5 Corollary Let $\{X_t,\mathcal{F}_t\}$ be a m.d. sequence, let $\{W_t\}$ be a sequence of positive $\mathcal{F}_{t-1}$-measurable r.v.s, and for some $p \geq 1$ let

$$D = \left\{\omega\colon \sum_{t=1}^{\infty} E(|X_t|^p|\mathcal{F}_{t-1})(\omega)/W_t^p(\omega) < \infty\right\} \in \mathcal{F}. \tag{20.12}$$

Also define, for any $r \geq p$,

$$D_1 = \left\{\omega\colon \sum_{t=1}^{\infty} P(|X_t| > W_t|\mathcal{F}_{t-1})(\omega) < \infty\right\} \in \mathcal{F} \tag{20.13}$$

$$D_2 = \left\{\omega\colon \sum_{t=1}^{\infty} |E(X_t 1_t^W|\mathcal{F}_{t-1})(\omega)|/W_t(\omega) < \infty\right\} \in \mathcal{F} \tag{20.14}$$

$$D_3 = \left\{\omega\colon \sum_{t=1}^{\infty} E(|X_t|^r 1_t^W|\mathcal{F}_{t-1})(\omega)/W_t^r(\omega) < \infty\right\} \in \mathcal{F}, \tag{20.15}$$

and let $D' = D_1 \cap D_2 \cap D_3$. Then $P(D - D') = 0$. In particular, If $P(D) = 1$ then $P(D') = 1$.

Proof It suffices to prove the three inequalities (20.9), (20.10), and (20.11) for the case of conditional expectations. Noting that $E(X_t|\mathcal{F}_{t-1}) = 0$ a.s. and using the fact that W_t is $\mathcal{F}_{t-1}$-measurable, all of these go through unchanged, except that the conditional modulus inequality **10.14** is used to get (20.14). It follows that almost every $\omega \in D$ is in D'. ∎

Another version of this theorem uses a different truncation, with the truncated variable chosen to be a continuous function of X_t; see **17.13** to appreciate why this variation might be useful.

20.6 Corollary Let $\{X_t\}_1^\infty$ be a zero-mean random sequence satisfying (20.5) for $p \geq 1$. Define

$$Y_t = X_t 1_t^a/a_t + (X_t/|X_t|)(1 - 1_t^a) = \begin{cases} X_t/a_t, & |X_t| \leq a_t \\ 1, & X_t > a_t \\ -1, & X_t < -a_t. \end{cases} \tag{20.16}$$

Then,

$$\sum_{t=1}^{\infty} |E(Y_t)| < \infty, \tag{20.17}$$