$$\sum_{t=1}^{\infty} E(X_t^2)/t^4 < \infty. \tag{20.37}$$

We cannot then rely upon $\bar{X}_n$ converging to zero, but (putting $a_t = t^2$) we can show that $n^{-2}\sum_{t=1}^n X_t = \bar{X}_n/n$ will do so.

The limitation of **20.10** is that it calls for square integrability. The next step is to use **20.4** to extend it to the class of cases that satisfy

$$\sum_{t=1}^{\infty} E|X_t|^p/a_t^p < \infty \tag{20.38}$$

for $1 \le p \le 2$, and some $\{a_t\} \uparrow \infty$. It is important to appreciate that (20.38) for $p < 2$ is not a *weaker* condition than for $p = 2$, and the latter does not imply the former. For contrast, consider $p = 1$. The Kronecker lemma applied to (20.38) implies that

$$a_n^{-1}\sum_{t=1}^{n} E|X_t| \longrightarrow 0. \tag{20.39}$$

For $a_n \sim n$, such a sequence has got to be zero or very close to it most of the time. In fact, there is a trivially direct proof of convergence. Note that

$$\begin{aligned} E\left(\lim_{n\to\infty} a_n^{-1}|S_n|\right) &\le E\left(\lim_{n\to\infty} a_n^{-1}\sum_{t=1}^{n}|X_t|\right) \\ &= \lim_{n\to\infty} a_n^{-1}\sum_{t=1}^{n} E|X_t| = 0, \end{aligned} \tag{20.40}$$

where the first equality follows by the dominated convergence theorem (**4.12**) with bounding function $g = \sum_{t=1}^{\infty}|X_t|/a_t$, where $E(g)$ is finite by assumption. For any random variable X, $E|X| = 0$ if and only if $X = 0$ a.s.. Nothing more is needed to show that S_n/a_n converges, regardless of other conditions.

Thus, having latitude in the value of p for which the theorem may hold is really a matter of being able to trade off the existence of absolute moments against the rate of damping necessary to make them summable. We may meet interesting cases in which (20.38) holds for $p < 2$ only rarely, but since this extension is available at small extra cost in complexity, it makes sense to take advantage of it.

20.11 Theorem If $\{X_t,\mathcal{F}_t\}_1^{\infty}$ is a m.d. sequence satisfying (20.38) for $1 \le p \le 2$, $S_n/a_n \xrightarrow{as} 0$.

Proof Let $Y_t = 1_{\{|X_t|\le a_t\}}X_t$, and note that $\{X_t\}$ and $\{Y_t\}$ are equivalent under (20.38), by **20.4**. Y_t is also $\mathcal{F}_t$-measurable, and hence the centred sequence $\{Z_t,\mathcal{F}_t\}$, where $Z_t = Y_t - E(Y_t|\mathcal{F}_{t-1})$, is a m.d. Now,

$$\begin{aligned} E(Z_t^2) &= E\big(E(Z_t^2|\mathcal{F}_{t-1})\big) \\ &= E\big(E(Y_t^2|\mathcal{F}_{t-1}) - E(Y_t|\mathcal{F}_{t-1})^2\big) \end{aligned}$$