turns out that neither approach dominates the other in terms of permissable conditions. We begin with the simplest of the arguments, the straightforward extension of **20.18**.

20.19 Theorem Let a sequence $\{X_t\}_{-\infty}^{\infty}$ with means $\{\mu_t\}_{-\infty}^{\infty}$ be L_p-NED of size $-b$, for $1 \le p \le 2$, with constants $d_t \ll \|X_t - \mu_t\|_p$, on a (possibly vector-valued) sequence $\{\boldsymbol{V}_t\}_{-\infty}^{\infty}$ which is α-mixing or ϕ-mixing of size $-a$. If

$$\sum_{t=1}^{\infty} \|(X_t - \mu_t)/a_t\|_q^{\xi} < \infty \tag{20.83}$$

for $q \ge p$ and $q > 1$, where

$$\xi = \min\left\{\frac{2qb + 2(q-1)}{(1+q)b + 2(q-1)}, \frac{2q(a+1)}{(1+q)a + 2q}\right\}, \tag{20.84}$$

then $a_n^{-1}\sum_{t=1}^n (X_t - \mu_t) \xrightarrow{as} 0$.

Proof By **17.5**, $\{X_t - \mu_t\}$ is a L_1-mixingale of size $-\min\{b,\ a(1-1/q)\}$ with respect to constants $\{c_t\}$, with $c_t \ll \|X_t - \mu_t\|_q$. This is by **17.5**(i) in the α-mixing case and by **17.5**(ii) in the ϕ-mixing case. The theorem follows by **20.18**, after substituting for φ_0 in (20.74) and simplifying. ∎

This permits arbitrary mixing and NED sizes and arbitrary moment restrictions, so long as (20.83) holds with ξ arbitrarily close to 1. By letting $b \to \infty$ one obtains a result for mixing sequences, and by letting $a \to \infty$ a result for sequences that are L_p-NED on an independent underlying process. Interestingly, in each of these special cases ξ ranges over the interval $(1, 2q/(q+1))$ as the mixing/L_p-NED size is allowed to range from zero to $-\infty$.

By contrast, a result based on **20.15** would be needed if we could claim only square-summability of the sequence $\{\|(X_t - \mu_t)/a_t\|_p\}$ for finite p; this rules out (20.83) for any choices of a and b. The first of these results comes directly by applying **17.5**.

20.20 Theorem For real numbers b, p and r, let a sequence $\{X_t\}_{-\infty}^{\infty}$ with means $\{\mu_t\}_{-\infty}^{\infty}$ be L_p-NED of size $-b$ on a sequence $\{\boldsymbol{V}_t\}_{-\infty}^{\infty}$, with constants $d_t \ll \|X_t - \mu_t\|_p$. For a positive constant sequence $\{a_t\} \uparrow \infty$, let $\{(X_t - \mu_t)/a_t\}_1^{\infty}$ be uniformly L_r-bounded, and let

$$\sum_{t=1}^{\infty} \|(X_t - \mu_t)/a_t\|_r^p < \infty. \tag{20.85}$$

Then $a_n^{-1}\sum_{t=1}^n (X_t - \mu_t) \xrightarrow{as} 0$ in each of the following cases:

(i) $b = \frac{1}{2}$, $p = 2$, $r > 2$, $\{\boldsymbol{V}_t\}$ is α-mixing of size $-r/(r-2)$;
(ii) $b = 1$, $1 < p < 2$, $r > p$, $\{\boldsymbol{V}_t\}$ is α-mixing of size $-pr/(r-p)$;
(iii) $b = \frac{1}{2}$, $p = 2$, $r \ge 2$, $\{\boldsymbol{V}_t\}$ is ϕ-mixing of size $-r/2(r-1)$;
(iv) $b = 1$, $1 < p < 2$, $r \ge p$, $\{\boldsymbol{V}_t\}$ is ϕ-mixing of size $-r/(r-1)$.

Proof By **17.5**, conditions (i)–(iv) are all sufficient for $\{(X_t - \mu_t)/a_t, \mathcal{F}_t\}$ to be an L_p-mixingale of size $-b$, where $\mathcal{F}_t = \sigma(\boldsymbol{V}_s, s \le t)$. The mixingale constants are $c_t/a_t \ll \max\{d_t, \|X_t - \mu_t\|_r\}/a_t = \|X_t - \mu_t\|_r/a_t$. The theorem follows by **20.16**. ■

As an example, let X_t be L_2-bounded, and be L_2-NED of size $-\frac{1}{2}$ on a ϕ-mixing process of size -1. Summability of the terms $\mathrm{Var}(X_t/a_t)$ is sufficient by (20.85). The same numbers yield $\xi = \frac{8}{7}$ on putting $q = 2$, $b = \frac{1}{2}$ and $a = 1$ in (20.84), which is not far from requiring summability of the L_2-norms.

However, this theorem requires L_r-boundedness, which if r is small constrains the permitted mixing size, as well as offering poor NED size characteristics for cases with $p < 2$. It can be improved upon in these situations by introducing a truncation argument. The third of our strong laws is the following.

20.21 Theorem Let a sequence $\{X_t\}_{-\infty}^{\infty}$ with means $\{\mu_t\}_{-\infty}^{\infty}$ be L_p-NED of size $-1/p$ for $1 \le p \le 2$, with constants $d_t \ll \|X_t - \mu_t\|_p$, on a sequence $\{\boldsymbol{V}_t\}_{-\infty}^{\infty}$ which is either

(i) α-mixing of size $-r/(r-2)$ for $r > 2$ or

(ii) ϕ-mixing of size $-r/2(r-1)$ for $r > 1$, and $r \ge p$;

and for q with $p \le q \le r$ and a constant positive sequence $\{a_t\} \uparrow \infty$, let

$$\sum_{t=1}^{\infty} \|(X_t - \mu_t)/a_t\|_q^{\min\{p, 2q/r\}} < \infty; \tag{20.86}$$

then $a_n^{-1}\sum_{t=1}^n (X_t - \mu_t) \xrightarrow{as} 0$. □

Note the different roles of the three constants specified in the conditions. p controls the size of the NED numbers, q is the minimum order of moment required to exist, and r controls the mixing of $\{\boldsymbol{V}_t\}$. The distribution of X_t does not otherwise depend on r.

Proof The strategy is to show that there is a sequence equivalent to $\{(X_t - \mu_t)/a_t\}$, and satisfying the conditions of **20.15**(i). As in **20.6**, let

$$Y_t = (X_t - \mu_t)1_t^a/a_t \pm (1 - 1_t^a), \tag{20.87}$$

where $1_t^a = 1_{\{|X_t - \mu_t| \le a_t\}}$ and '$\pm$' denotes '+' if $X_t > \mu_t$, '$-$' otherwise. Note that $\{(X_t - \mu_t)/a_t\}$ is L_p-NED with constants d_t/a_t, and Y_t is a continuous function of $(X_t - \mu_t)/a_t$ with $|Y_t| \le 1$ a.s. Applying **17.13** shows that Y_t is L_2-NED on $\{\boldsymbol{V}_t\}$ of size $-\frac{1}{2}$ with constants $2^{1-p/2}(d_t/a_t)^{p/2}$. Since $\|Y_t\|_r < \infty$ for every finite r, it further follows by **17.5** that if $\mathcal{F}_t = \sigma(\boldsymbol{V}_{t-s}, \text{s} \ge 0)$, $\{Y_t - E(Y_t), \mathcal{F}_t\}_0^{\infty}$ is an L_2-mixingale of size $-\frac{1}{2}$ with constants

$$c_t \ll \max\{(d_t/a_t)^{p/2}, \|Y_t\|_r\}. \tag{20.88}$$

Here, $d_t \le 2\|X_t - \mu_t\|_q$ for any $q \ge p$, and $\|Y_t\|_r \le 2(\|X_t - \mu_t\|_q/a_t)^{q/r}$ for any $q \le r$ by the second inequality of (20.24). Condition (20.86) is therefore sufficient for the sequence $\{c_t^2\}$ to be summable, and $\{Y_t - E(Y_t)\}$ satisfies the conditions of **20.15**(i). We can conclude that $\sum_{t=1}^n (Y_t - E(Y_t)) \xrightarrow{as} S_1$, where S_1 is some random variable.

According to **20.6**, condition (20.86) is sufficient for $\sum_{t=1}^{\infty}|E(Y_t)| < \infty$. The series $\sum_{t=1}^n E(Y_t)$ therefore converges to a finite limit by **2.24**, say $\sum_{t=1}^{\infty} E(Y_t) =$

C_1, and

$$\sum_{t=1}^{n} Y_t \xrightarrow{as} S_1 + C_1. \tag{20.89}$$

Inequalities (20.22) and (20.6) further imply that Y_t and $(X_t - \mu_t)/a_t$ are equivalent sequences, and hence

$$\sum_{t=1}^{n} \big((X_t - \mu_t)/a_t - Y_t\big) \xrightarrow{as} S_2, \tag{20.90}$$

where S_2 is another random variable, by **20.3**. We conclude that

$$\sum_{t=1}^{n} (X_t - \mu_t)/a_t \xrightarrow{as} S_1 + S_2 + C_1 = S_3, \tag{20.91}$$

say. It follows by Kronecker's lemma that $a_n^{-1}\sum_{t=1}^{n}(X_t - \mu_t) \xrightarrow{as} 0$, the required conclusion. ■

Here is a final, more specialized, result. The linear function of martingale differences with summable coefficients is a case of particular interest since it unifies our two approaches to the strong law.

20.22 Theorem Let $X_t = \sum_{j=-\infty}^{\infty}\theta_j U_{t-j}$ where $\{U_t\}$ is a uniformly L_p-bounded m.d. sequence with $p > 1$, and $\sum_{j=-\infty}^{\infty}|\theta_j| < \infty$. Then

$$\frac{1}{n}\sum_{t=1}^{n} X_t \xrightarrow{as} 0. \tag{20.92}$$

Proof Letting $Y_t = X_t/t$ and $\mathcal{F}_t = \sigma(U_s,\ s \le t)$, the sequence $\{Y_t, \mathcal{F}_t\}_1^\infty$ is a L_p-mixingale with $c_t \ll 1/t$, and arbitrary size. It was shown in **16.12** that the maximal inequality (20.56) holds for this case. Application of the convergence lemma and Kronecker's lemma lead directly to the result. Alternatively, apply **20.18** to X_t with $a_t = t$. ■

In these results, four features summarize the relevant characteristics of the stochastic process: the order of existing moments, the summability characteristics of the moments, and the sizes of the mixing and near-epoch dependence numbers. The way in which the currently available theorems trade off these features suggests that some unification should be possible. The McLeish-style argument is revealed by de Jong's approach to be excessively restrictive with respect to the dependence conditions it imposes, whereas the tough summability conditions the latter's theorem requires may also be an artefact of the method adopted. The repertoire of dependent strong laws is currently being extended (de Jong, 1994) in work as yet too recent for incorporation in this book.