then G_n is s.e. □

Condition (21.61) is an interesting Cesàro-sum variation on uniform integrability, and actual uniform integrability of $\{d_t\}$ is sufficient, although not necessary. Condition (a) is a domination condition, while condition (b) is called by Andrews *termwise stochastic equicontinuity*.

Proof Given $\varepsilon > 0$, choose M such that $\limsup_{n\to\infty} n^{-1}\sum_{t=1}^{n} E(2d_t 1_{\{2d_t > M\}}) < \frac{1}{6}\varepsilon^2$, and then δ such that

$$\limsup_{n\to\infty} \frac{1}{n}\sum_{t=1}^{n} P\big(w(q_t,\delta) > \tfrac{1}{6}\varepsilon^2\big) < \tfrac{1}{6}M^{-1}\varepsilon^2. \tag{21.63}$$

The first thing to note is that

$$\begin{aligned} w((q_t - E(q_t)),\delta) &\le w(q_t,\delta) + w(E(q_t),\delta) \\ &\le w(q_t,\delta) + E(w(q_t,\delta)), \end{aligned} \tag{21.64}$$

where the last inequality is an application of **21.3**. Applying (21.64) and using Markov's inequality,

$$\begin{aligned} P\big(w(G_n,\delta) > \varepsilon\big) &\le P\left(\frac{1}{n}\sum_{t=1}^{n} w(q_t - E(q_t),\delta) > \varepsilon\right) \\ &\le P\left(\frac{1}{n}\sum_{t=1}^{n} \big(w(q_t,\delta) + E(w(q_t,\delta))\big) > \varepsilon\right) \\ &\le \frac{2}{n\varepsilon}\sum_{t=1}^{n} E\big(w(q_t,\delta)\big) \\ &= \frac{2}{n\varepsilon}\sum_{t=1}^{n} E\big[w(q_t,\delta)\big(1_{\{w(q_t,\delta)\le\varepsilon^2/6\}} \\ &\qquad + 1_{\{\varepsilon^2/6 < w(q_t,\delta)\le M\}} + 1_{\{w(q_t,\delta) > M\}}\big)\big], \end{aligned} \tag{21.65}$$

where the indicator functions in the last member add up to 1. Using the fact that $w(q_t,\delta) \le 2d_t$, and hence $\{w(q_t,\delta) > M\} \subseteq \{2d_t > M\}$, and taking the limsup, we now obtain

$$\begin{aligned} \limsup_{n\to\infty} P\big(w(G_n,\delta) > \varepsilon\big) &\le \frac{2}{\varepsilon}\Bigg[\frac{\varepsilon^2}{6} + M\limsup_{n\to\infty}\frac{1}{n}\sum_{t=1}^{n} P\left(w(q_t,\delta) > \frac{\varepsilon^2}{6}\right) \\ &\qquad + \limsup_{n\to\infty}\frac{1}{n}\sum_{t=1}^{n} E\big(2d_t 1_{\{2d_t > M\}}\big)\Bigg] \\ &\le \varepsilon, \end{aligned} \tag{21.66}$$

in view of the values chosen for M and δ. ■

Clearly, whether condition **21.12**(a) is satisfied depends on both the distribution of X_t and functional form of $q_t(.)$. But something relatively general can be said about termwise s.e. (condition **21.12**(b)). Assume, following Pötscher and Prucha (1989), that

$$q_t(x,\theta) = \sum_{k=1}^{p} r_{kt}(x)s_{kt}(x,\theta), \tag{21.67}$$

where r_{kt}: $\mathbb{X} \to \mathbb{R}$, and $s_{kt}(.,\theta)$: $\mathbb{X} \to \mathbb{R}$ for fixed θ, are $\mathcal{X}/\mathcal{B}$-measurable functions. The idea here is that we can be more liberal in the behaviour allowed to the factors r_{kt} as functions of X_t than to the factors s_{kt}; discontinuities are permitted, for example. To be exact, we shall be content to have the r_{kt} asymptotically L_1-bounded in Cesàro mean:

$$\limsup_{n\to\infty} \frac{1}{n}\sum_{t=1}^{n} E|r_{kt}(X_t)| \leq B < \infty,\ k = 1,\dots,p. \tag{21.68}$$

As for the factors $s_{kt}(x,\theta)$, we need these to be asymptotically equicontinuous for a sufficiently large set of x values. Assume there is a sequence of sets $\{K_m \in \mathcal{X},$ $m = 1,2,\dots\}$, such that

$$\limsup_{n\to\infty} \frac{1}{n}\sum_{t=1}^{n} \mu_t(K_m^c) \to 0 \text{ as } m \to \infty, \tag{21.69}$$

and that for each $m \geq 1$ and $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\limsup_{t\to\infty} \sup_{x\in K_m} w(s_{kt}(x,.),\delta) < \varepsilon,\ k = 1,\dots,p. \tag{21.70}$$

Notice that (21.70) is a *non*stochastic equicontinuity condition, but under condition (21.69) it holds (as one might say) 'almost surely, on average' when the r.v. X_t is substituted into the formula.

These conditions suffice to give termwise s.e., and hence can be used to prove s.e. of G_n by application of **21.12**.

21.13 Theorem If $q_t(X_t,\theta)$ is defined by (21.67), and (21.68), (21.69), and (21.70) hold, then for every $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\limsup_{n\to\infty} \frac{1}{n}\sum_{t=1}^{n} P\big(w(q_t,\delta) > \varepsilon\big) < \varepsilon. \tag{21.71}$$

Proof Fix $\varepsilon > 0$, and first note that

$$P\big(w(q_t,\delta) > \varepsilon\big) \leq P\left(\sum_{k=1}^{p} |r_{kt}|\, w(s_{kt},\delta) > \varepsilon\right)$$

$$\leq P\left(\bigcup_{k=1}^{p} \{|r_{kt}|\, w(s_{kt},\delta) > \varepsilon/p\}\right)$$

$$\leq \sum_{k=1}^{p} P\big(|r_{kt}|w(s_{kt},\delta) > \varepsilon/p\big)$$

$$\leq \sum_{k=1}^{p} \left[P\left(|r_{kt}|w(s_{kt},\delta)1_{K_m} > \frac{\varepsilon}{2p}\right) + P\left(|r_{kt}|w(s_{kt},\delta)1_{K_m^c} > \frac{\varepsilon}{2p}\right)\right]. \tag{21.72}$$

Consider any one of these p terms. Choose m large enough that

$$\limsup_{n\to\infty} \frac{1}{n}\sum_{t=1}^{n} \mu_t(K_m^c) < \frac{\varepsilon}{2p}. \tag{21.73}$$

For this m there exist, by (21.70), $\delta > 0$ and $t_0 \geq 1$ such that

$$\sup_{x \in K_m} w(s_{kt}(x,.),\delta) < \frac{\varepsilon^2}{4Bp^2} \tag{21.74}$$

for $t > t_0$. Therefore, applying the Markov inequality and then (21.68),

$$\limsup_{n\to\infty} \frac{1}{n}\sum_{t=1}^{n} P\left(|r_{kt}|w(s_{kt},\delta)1_{K_m} > \frac{\varepsilon}{2p}\right)$$

$$\leq \limsup_{n\to\infty} \frac{1}{n}\left(t_0 + \sum_{t=t_0+1}^{n} P\left(|r_{kt}|\frac{\varepsilon^2}{4Bp^2} > \frac{\varepsilon}{2p}\right)\right)$$

$$\leq \limsup_{n\to\infty} \frac{1}{n}\sum_{t=t_0+1}^{n} E|r_{kt}|\frac{\varepsilon}{2Bp}$$

$$\leq \frac{\varepsilon}{2p}, \tag{21.75}$$

whereas by (21.73),

$$\limsup_{n\to\infty} \frac{1}{n}\sum_{t=1}^{n} P\left(|r_{kt}|w(s_{kt},\delta)1_{K_m^c} > \frac{\varepsilon}{2p}\right) \leq \limsup_{n\to\infty} \frac{1}{n}\sum_{t=1}^{n} P\big(X_t \notin K_m\big)$$

$$\leq \frac{\varepsilon}{2p}. \tag{21.76}$$

Substituting these bounds into (21.72) yields the result. ■