$$1 + i\lambda Z_{ni} = \exp\{i\lambda Z_{ni}\}\exp\{\tfrac{1}{2}\lambda^2 Z_{ni}^2 + r(i\lambda Z_{ni})\}, \tag{24.2}$$

where the remainder satisfies $|r(x)| \le |x|^3$ for $|x| < 1$. Multiplying up the terms for $i = 1,...,r_n$ yields

$$\exp\{i\lambda S_{r_n}\} = T_{r_n}U_{r_n},$$

where $T_{r_n}$ is defined in (24.1) and

$$U_{r_n} = \exp\left\{-\tfrac{1}{2}\lambda^2\sum_{i=1}^{r_n} Z_{ni}^2 - \sum_{i=1}^{r_n} r(i\lambda Z_{ni})\right\}. \tag{24.3}$$

Taking expectations produces

$$\phi_{S_{r_n}}(\lambda) = E(T_{r_n}U_{r_n}) = e^{-\lambda^2/2}E(T_{r_n}) + E\big(T_{r_n}\big(U_{r_n} - e^{-\lambda^2/2}\big)\big), \tag{24.4}$$

so given condition (b) of the theorem, $\phi_{S_{r_n}}(\lambda) \to e^{-\lambda^2/2}$ if

$$\lim_{n\to\infty} E\left|T_{r_n}\big(U_{r_n} - e^{-\lambda^2/2}\big)\right| = 0. \tag{24.5}$$

The sequence

$$T_{r_n}\big(U_{r_n} - e^{-\lambda^2/2}\big) = \exp\{i\lambda S_{r_n}\} - T_{r_n}e^{-\lambda^2/2} \tag{24.6}$$

is uniformly integrable in view of condition (a), the first term on the right-hand side having unit modulus. So in view of **18.14**, it suffices to show that

$$\operatorname*{plim}_{n\to\infty} T_{r_n}(U_{r_n} - e^{-\lambda^2/2}) = 0. \tag{24.7}$$

Since $T_{r_n}$ is clearly $O_p(1)$, the problem reduces, by **18.12**, to showing that $\text{plim}_{n\to\infty}U_{r_n} = e^{-\lambda^2/2}$, and for this in turn it suffices, by condition (c), if

$$\operatorname*{plim}_{n\to\infty} \sum_{i=1}^{r_n} r(i\lambda Z_{ni}) = 0. \tag{24.8}$$

But this follows from conditions (c) and (d), and **18.12**, given that the inequality

$$\left|\sum_{i=1}^{r_n} r(i\lambda Z_{ni})\right| \le |\lambda|^3\sum_{i=1}^{r_n}|Z_{ni}|^3 \le |\lambda|^3\left(\max_{1\le i\le r_n}|Z_{ni}|\right)\sum_{i=1}^{r_n} Z_{ni}^2 \tag{24.9}$$

holds with probability approaching 1 by condition (d), as $n \to \infty$. ■

It is instructive to compare this proof with that of the Lindeberg theorem. A different series approximation of the ch.f. is used, and the assumption from independence, that

$$\phi_{S_{r_n}} = \textstyle\prod_i \phi_{Z_{ni}},$$

is avoided. Of course, we have yet to show that conditions **24.1**(a) and **24.1**(b) hold under convenient and plausible assumptions about the sequence. The rest of