$$\leq \sum_{j=1}^{b_n-1}\left(E\left|E\big(W_{ni}(j,0)\,|\,\mathcal{F}_{-\infty}^{(i-p)b_n}\big)\right| + 2\sum_{k=1}^{b_n-j}E\left|E\big(W_{ni}(j,k)\,|\,\mathcal{F}_{-\infty}^{(i-p)b_n}\big)\right|\right)$$

$$+ E\left|E\big(W_{ni}(b_n,0)\,|\,\mathcal{F}_{-\infty}^{(i-p)b_n}\big)\right|$$

$$\ll M_{ni}^2\left(\sum_{j=1}^{b_n-1}\big(1+2(b_n-j)\big)\xi_{(p-1)b_n+j}+\xi_{pb_n}\right), \tag{24.67}$$

and similarly,

$$E\left|Z_{ni}^2 - E\big(Z_{ni}^2\,|\,\mathcal{F}_{-\infty}^{(i+p)b_n}\big)\right|$$

$$\ll M_{ni}^2\left[\sum_{j=1}^{b_n-1}\left(\xi_{(p+1)b_n-j}+2\sum_{k=1}^{b_n-j}\xi_{(p+1)b_n-j-k}\right)+\xi_{pb_n}\right], \tag{24.68}$$

where $\xi_j = O(j^{-1-\delta})$ for $\delta > 0$. Write, formally, $a_{ni}\psi_p^*$ to denote the larger of the two majorant expressions in (24.67) and (24.68), such that $\psi_p^* \longrightarrow 0$ and a_{ni} is fixed by setting $\psi_0^* = 1$. Evaluating (24.67) at $p = 1$ and (24.68) at $p = 0$ respectively gives

$$E\left|E\big(Z_{ni}^2 - E(Z_{ni}^2)\,|\,\mathcal{F}_{-\infty}^{(i-1)b_n}\big)\right| \ll M_{ni}^2\left(\sum_{j=1}^{b_n-1}\big(1+2(b_n-j)\big)j^{-1-\delta}+b_n^{-1-\delta}\right)$$

$$\ll M_{ni}^2 b_n, \tag{24.69}$$

and also, putting $j' = b_n - j$ and $k' = j' - k$,

$$E\left|Z_{ni}^2 - E\big(Z_{ni}^2\,|\,\mathcal{F}_{-\infty}^{ib_n}\big)\right| \ll M_{ni}^2\left[\sum_{j'=1}^{b_n-1}\left(j'^{-1-\delta}+2\sum_{k'=0}^{j'-1}k'^{-1-\delta}\right)+1\right] \tag{24.70}$$

$$\ll M_{ni}^2 b_n.$$

Hence, $a_{ni} = BM_{ni}^2 b_n$ for some finite constant B. Since

$$\sum_{i=1}^{r_n} M_{ni}^4 \leq \max_{1\leq i\leq r_n+1} M_{ni}^2 \sum_{i=1}^{r_n} M_{ni}^2 = o(b_n^{-2}) \tag{24.71}$$

in view of (24.27) and (24.25), these constants satisfy conditions **19.11**(b) and (c). And since $Z_{ni}^2/b_nM_{ni}^2 \leq Z_{ni}^2/\nu_{ni}^2$ where Z_{ni}^2/ν_{ni}^2 is uniformly integrable, they also satisfy condition **19.11**(a). It follows that $A_n \xrightarrow{L_1} 0$, and the proof is complete. ■

This brings us to the final step in the argument, establishing the asymptotic m.d. property of the Bernstein blocks.

24.19 Theorem Under **24.6**(a)–(d), $\lim_{n\to\infty} E(\tilde{T}_{r_n}) = 1$.

Proof Applying (24.17),