In the proof of the CLT the construction of the Bernstein blocks was purely conceptual, but we might consider the option of actually computing them. The sum of squares of the unweighted blocks, $\hat{s}_{n1}^2 = \sum_{i=1}^{r_n} Z_{ni}^{*2}$, is consistent for s_n^2 in the sense that

$$\frac{\hat{s}_{n1}^2}{s_n^2} = \sum_{i=1}^{r_n} Z_{ni}^2 \xrightarrow{pr} 1, \tag{25.6}$$

according to **24.17** and **24.18**. An important rider to this proposal is that **24.17** and **24.18** were proved using only (24.25) and (24.27), so that, as noted previously, any $\alpha \in (0,1)$ will serve to construct the blocks. In the context of the conditions of **24.12** at least, the only constraint imposed by **24.6**(d) is represented by (24.39), which puts a possible lower bound on α in the case of decreasing variances ($\gamma > 0$), but no upper bound strictly less than 1. It is sufficient for consistency if b_n goes to infinity at a positive rate, and we are not bound to use the α that satisfies the conditions of the CLT to construct the blocks in (25.6).

But although consistent, $\hat{s}_{n1}^2$ is not the obvious choice of estimator. It would be more natural to follow authors such as Newey and West (1987) and Gallant and White (1988), *inter alia*, who consider estimators based on all the cross-products X_tX_{t-k} for $t = k+1,\ldots,n$ and $k = 0,\ldots,b_n$. In terms of the array represention of Fig. 24.1, these are the elements in the diagonal *band* of width $2b_n$, rather than the diagonal blocks only. (In this context, b_n is referred to as the *band width*.) The simplest such estimator is

$$\hat{s}_{n2}^2 = \sum_{t=1}^{n} X_t^2 + 2\sum_{k=1}^{b_n}\sum_{t=k+1}^{n} X_tX_{t-k}. \tag{25.7}$$

25.3 Theorem Under the conditions of **24.6**, applied to X_t/s_n, $\hat{s}_{n2}^2/s_n^2 \xrightarrow{pr} 1$.

Proof Let $X_{nt} = X_t/s_n$ in (24.53), so that $s_n^2A_n$ denotes the same sum constructed from the X_t in place of the X_{nt}. The difference between A_n and $\big(\hat{s}_{n2}^2 - E(\hat{s}_{n2}^2)\big)/s_n^2$ is the quantity

$$A_n^* = \frac{1}{s_n^2}\left[2\sum_{i=2}^{r_n}\left(\sum_{t=(i-1)b_n+1}^{ib_n}\ \sum_{k=t-(i-1)b_n,\ t<ib_n}^{b_n-1}(X_tX_{t-k} - \sigma_{t,t-k})\right)\right.$$
$$\left. + \sum_{t=r_nb_n+1}^{n}\left((X_t^2 - \sigma_t^2) + 2\sum_{k=1}^{b_n-1}(X_tX_{t-k} - \sigma_{t,t-k})\right)\right] \tag{25.8}$$

The components of this sum correspond to the $r_n - 1$ 'triangles' which separate the diagonal blocks in Fig. 24.1, each containing $\frac{1}{2}b_n(b_n-1)$ terms, plus the terms from the lower-right corner blocks. Reasoning closely analogous to the proof of **24.18** shows that $A_n^* \xrightarrow{L_1} 0$. The sums of the corresponding covariances converge absolutely to 0 by **24.17**, since they are components of B_n' in (24.54), and it follows that

$$\frac{|\hat{s}_{n2}^2 - \hat{s}_{n1}^2|}{s_n^2} \xrightarrow{pr} 0.$$

The theorem therefore follows from **24.18**. ■

Since this estimator uses sample data which are discarded in $\hat{s}_{n1}^2$, there are informal reasons for preferring it in small samples. But there is a problem in that (the chosen notation notwithstanding) $\hat{s}_{n2}^2$ is *not* a square, and not always non-negative except in the limit. This difficulty can be overcome by inserting fixed weights in the sum, as in

$$\hat{s}_{n3}^2 = \sum_{t=1}^{n} X_t^2 + 2\sum_{k=1}^{b_n} w_{nk} \sum_{t=k+1}^{n} X_t X_{t-k}. \tag{25.9}$$

Suppose $w_{nk} \longrightarrow 1$ as $n \longrightarrow \infty$ for every $k \le K$, for every fixed $K < \infty$. Then

$$\frac{|\hat{s}_{n2}^2 - \hat{s}_{n3}^2|}{s_n^2} \le \frac{2}{s_n^2}\left|\sum_{k=1}^{K}(1 - w_{nk})\sum_{t=k+1}^{n} X_t X_{t-k}\right| + r_n(K) \xrightarrow{pr} r(K), \tag{25.10}$$

where $r(K) = \operatorname{plim}_{n\to\infty} r_n(K)$, and

$$r_n(K) = \frac{2}{s_n^2}\left|\sum_{k=K+1}^{b_n}(1 - w_{nk})\sum_{t=k+1}^{n} X_t X_{t-k}\right|. \tag{25.11}$$

Since $r(K)$ can be made as small as desired by taking K large enough, in view of **25.3** and **24.18**, $\hat{s}_{n3}^2$ is consistent when the weights have this property. It remains to see if they can also be chosen in such a way that (25.9) is a sum of squares.

Following Gallant and White (1988), choose $b_n + 1$ real numbers $a_{n1},\ldots,a_{n,b_n+1}$, satisfying $\sum_{j=1}^{b_n+1} a_{nj}^2 = 1$, and consider the $n + b_n$ variables

$$\begin{aligned}
Y_{n1} &= a_1 X_1, \\
Y_{n2} &= a_{n1}X_2 + a_{n2}X_1, \\
&\ldots \\
Y_{n,b_n+1} &= a_{n1}X_{b_n+1} + \ldots + a_{n,b_n+1}X_1, \\
Y_{n,b_n+2} &= a_{n1}X_{b_n+2} + \ldots + a_{n,b_n+1}X_2, \\
&\ldots \\
Y_{nn} &= a_{n1}X_n + \ldots + a_{n,b_n+1}X_{n-b_n}, \\
Y_{n,n+1} &= a_{n2}X_n + \ldots + a_{nb_n}X_{n-b_n+1}, \\
&\ldots \\
Y_{n,n+b_n} &= a_{nb_n}X_n.
\end{aligned}$$

Observe that

$$\begin{aligned}
\sum_{t=1}^{n+b_n} Y_{nt}^2 &= \left(\sum_{j=1}^{b_n+1} a_{nj}^2\right)\sum_{t=1}^{n} X_t^2 + 2\left(\sum_{j=2}^{bn+1} a_{nj}a_{n,j-1}\right)\sum_{t=2}^{n} X_t X_{t-1} \\
&\quad + 2\left(\sum_{j=3}^{bn+1} a_{nj}a_{n,j-2}\right)\sum_{t=3}^{n} X_t X_{t-2} + \ldots + 2a_{n,b_n+1}a_{n1}\sum_{t=b_n+1}^{n} X_t X_{t-b_n},
\end{aligned} \tag{25.12}$$