(a) $W(x(0) = 0) = 1$;

(b) $W(x(t) \leq a) = \dfrac{1}{\sqrt{2\pi t}}\displaystyle\int_{-\infty}^{a} e^{-\xi^2/2t}d\xi,\ 0 < t \leq 1$;

(c) for every partition $\{t_1,\dots,t_k\}$ of $[0,1]$, the increments $x(t_1) - x(t_0)$, $x(t_2) - x(t_1),\dots,x(t_k) - x(t_{k-1})$ are totally independent. □

Parts (a) and (b) of the definition give the marginal distributions of the coordinate functions, while condition (c) fixes their joint distribution. Any finite collection of process coordinates $\{x(t_i),\ i = 1,\dots,k\}$ has the multivariate Gaussian distribution, with $x(t_j) \sim N(0,t_j)$, and $E(x(t_j)x(t_{j'})) = \min\{t_j,t_{j'}\}$. Hence, $x(t_1) - x(t_2) \sim N(0, |t_1 - t_2|)$, which agrees with the requirements of continuity. This full specification of the finite-dimensional distributions suffices to define a unique measure on $(C,\mathcal{B}_C)$. This does not amount to proving that such a measure exists, but we shall show this below; see **27.15**.

W may equally well be defined on the interval $[0,b)$ for any $b > 0$, including $b = \infty$, but the cases with $b \neq 1$ will not usually concern us here.

A random element distributed according to W is called a Wiener process or a *Brownian motion* process. The latter term refers to the use of this p.m. as a mathematical model of the random movements of pollen grains suspended in water resulting from thermal agitation of the water molecules, first observed by the botanist Robert Brown.[27] In practice, the terms Wiener process and Brownian motion tend to be used synonymously. The symbol W conventionally stands for the p.m., and we also follow convention in using the symbol B to denote a random element from the derived probability space $(C,\mathcal{B}_C,W)$. In terms of the underlying probability space $(\Omega,\mathcal{F},P)$ on which we assume $B\colon \Omega \mapsto C$ to be a $\mathcal{F}/\mathcal{B}_C$-measurable mapping, we have $W(E) = P(B \in E)$ for each set $E \in \mathcal{B}_C$.

The continuous graph of a random element of Brownian motion, $B(\omega)$ for $\omega \in \Omega$, is quite a remarkable object (see Fig. 27.5). It belongs to the class of geometrical forms named *fractals* (Mandelbrot 1983). These are curves possessing the property of *self-similarity*, meaning essentially that their appearance is invariant to scaling operations. It is straightforward to verify from the definition that if B is a Brownian motion so is B^*, where

$$B^*(\omega,t) = k^{-1/2}\big(B(\omega,s + kt) - B(\omega,s)\big) \tag{27.31}$$

for any $s \in [0,1)$ and $k \in (0,1 - s]$. Varying s and k can be thought of as 'zooming in' on the portion of the process from s to $s + k$.

The key property is the one contained in part (iii) of the definition, that of independent increments. A little thought is required to see what this means. In the definition, the points $t_1,\dots,t_k$ may be arbitrarily close together. Considering a pair of points t and $t + \Delta$, the increment $B(\omega,t + \Delta) - B(\omega,t)$ is Gaussian with variance Δ, and independent of $B(\omega,t)$. Symmetry of the Gaussian density implies that

$$P\big(\omega\colon (B(\omega,t + \Delta) - B(\omega,t))(B(\omega,t) - B(\omega,t - \Delta)) < 0\big) = \tfrac{1}{2}$$