processes having independent increments $X(t) - X(s) \sim N(\mu(t-s), \sigma^2|t-s|)$.

More elaborate generalizations of Brownian motion include the following.

27.8 Example Let $X(t) = B\big(t^{1+\beta}\big)$ for $-1 < \beta < \infty$. X is a Brownian motion which has been subjected to stretching and squeezing of the time domain. Like B, it is a.s. continuous with independent Gaussian increments. It can be thought of as the limit of a partial sum process whose increments have trending variance. Suppose $\xi_i(\omega) \sim N\big(0, (1+\beta)i^{\beta}\big)$, which means the variances are tending to 0 if $\beta < 0$, or to infinity if $\beta > 0$. Then $n^{-1-\beta}E\big(\sum_{i=1}^{[nt]}\xi_i\big)^2 \to t^{1+\beta}$, and

$$n^{-(1+\beta)/2}\sum_{i=1}^{[nt]}\xi_i(\omega) \to B(\omega, t^{1+\beta}) \text{ a.s. } \square \tag{27.36}$$

27.9 Example Let $X(t) = \theta(t)B(t)$ where θ: $[0,1] \mapsto \mathbb{R}$ is any continuous deterministic function, and B is a Brownian motion. For $s < t$,

$$X(t) - X(s) = \theta(t)(B(t) - B(s)) + (\theta(t) - \theta(s))B(s), \tag{27.37}$$

which means that the increments of this process, while Gaussian, are not independent. It can be thought of as the almost sure limit as $n \to \infty$ of a double partial sum process,

$$n^{-1/2}\sum_{i=1}^{[nt]}\left[\theta(i/n)\xi_i(\omega) + \big(\theta(i/n) - \theta((i-1)/n)\big)\sum_{j=1}^{i-1}\xi_j(\omega)\right], \tag{27.38}$$

where $\xi_i \sim N(0,1)$. $\square$

27.10 Example Letting B denote standard Brownian motion on $[0,\infty)$, define

$$X(t) = e^{-\beta t}B(e^{2\beta t}) \tag{27.39}$$

for fixed $\beta > 0$. This is a zero-mean Gaussian process, having dependent increments like **27.9**. The remarkable feature of this process is that it is stationary, with $X(t) \sim N(0,1)$ for all $t > 0$, and

$$E(X(t)X(s)) = e^{\beta(2\min\{t,s\} - t - s)} = e^{-\beta|t-s|}. \tag{27.40}$$

This is the *Ornstein-Uhlenbeck process*. $\square$

27.11 Example The *Brownian bridge* is the process $B^o \in C$ where

$$B^o(t) = B(t) - tB(1),\ t \in [0,1]. \tag{27.41}$$

This is a Brownian motion tied down at both ends, and has $E(B^o(t)B^o(s)) = \min\{t,s\} - ts$. A natural way to think about B^o is as the limit of the partial sums of a mean-deviation process, that is

$$B^o(t,\omega) = \lim_{n\to\infty} n^{-1/2}\sum_{i=1}^{[nt]}\left(\xi_i(\omega) - \frac{1}{n}\sum_{j=1}^{n}\xi_j(\omega)\right) \text{ a.s.} \tag{27.42}$$

where $\xi_i(\omega) \sim N(0,1)$. $\square$