$$\geq \max\left\{\mu_n(\{x\colon |x(0)| > M\}),\ \mu_n(\{x\colon w_x(\delta) \geq \varepsilon\})\right\}. \qquad (27.48)$$

Hence (27.45) and (27.46) hold, for all $n \in \mathbb{N}$.

Write $\mu^*(.)$ as shorthand for $\sup_{n\geq N}\mu_n(.)$. To prove sufficiency, consider for $k = 1,2,...$ the sets

$$A_k = \{x\colon w_x(\delta_k) < 1/k\}, \qquad (27.49)$$

where $\{\delta_k\}$ is a sequence chosen so that $\mu^*(A_k) > 1 - \theta/2^{k+1}$, for $\theta > 0$. This is possible by condition (b). Also set $B = \{x\colon |x(0)| \leq M\}$, where M is chosen so that $\mu^*(B) > 1 - \theta/2$, which is possible by condition (a). Then define a closed set $K = \left(\bigcap_{k=1}^{\infty} A_k \cap B\right)^{-}$, and note that conditions (27.16) and (27.17) hold for the case $A = K$. Hence by **27.5**, K is compact. But

$$\mu^*(K^c) \leq \mu^*\left(\bigcup_{k=1}^{\infty} A_k^c \cup B^c\right)$$

$$\leq \sum_{k=1}^{\infty} \mu^*(A_k^c) + \mu^*(B^c)$$

$$\leq 2\theta\sum_{k=1}^{\infty} 1/2^{k+2} + \theta/2 = \theta. \qquad (27.50)$$

This last inequality is to be read as $\sup_{n\geq N}\mu_n(K^c) \leq \theta$, or equivalently $\inf_{n\geq N}\mu_n(K) > 1 - \theta$. Since θ is arbitrary, and every individual μ_n is tight by **26.19**, in particular for $1 \leq n < N$, it follows that the sequence $\{\mu_n\}$ is uniformly tight. ■

The following lemma is a companion to the last result, supplying in conjunction with it a relatively primitive sufficient condition for uniform tightness.

27.13 Lemma (adapted from Billingsley 1968: th 8.3) Suppose that, for some $\delta \in (0,1)$ and $N \in \mathbb{N}$,

$$\sup_{0\leq t\leq 1-\delta} \mu_n\left(\left\{x\colon \sup_{t\leq s\leq t+\delta} |x(s) - x(t)| \geq \tfrac{1}{2}\varepsilon\right\}\right) \leq \tfrac{1}{2}\eta\delta \qquad (27.51)$$

for all $\varepsilon > 0$, $\eta > 0$ and $n \geq N$. Then, (27.46) holds in the same cases.

Proof Fixing δ, consider the partition $\{t_1,...,t_r\}$ of $[0,1]$, for $r = 1 + [1/\delta]$, where $t_i = i\delta$ for $i = 1,...,r-1$ and $t_r = 1$. Thus, for $\frac{1}{2} < \delta < 1$ we have $r = 2$ and the partition $\{\delta, 1\}$, for $\frac{1}{3} < \delta \leq \frac{1}{2}$ we have $r = 2$ and the partition $\{\delta, 2\delta, 1\}$, and so on. The width of these intervals is at most δ. A given interval $[t,t']$ with $|t' - t| \leq \delta$ must either lie within an interval of the partition, or at most overlap two adjoining intervals; it cannot span three or more. In the event that $|x(t') - x(t)| \geq \varepsilon$, x must change absolutely by at least $\frac{1}{2}\varepsilon$ in at least one of the interval(s) overlapping $[t,t']$, and the probability of the latter event is at least that of the former. In other words, considering all such intervals,