which sequences such as the one in (28.12) are not Cauchy sequences. Ingenious alternatives have been suggested by different authors. The following is due to Billingsley (1968), from which source the results of this section and the next are adapted.

Let Λ be the collection of homeomorphisms λ from [0,1] to [0,1] with $\lambda(0) = 0$ and $\lambda(1) = 1$, and satisfying

$$\|\lambda\| = \sup_{t\neq s}\left|\log\frac{\lambda(t)-\lambda(s)}{t-s}\right| < \infty. \tag{28.13}$$

Here, $\|\lambda\|: \Lambda \mapsto \mathbb{R}^+$ is a functional measuring the maximum deviation of the gradient of λ from 1, so that in particular $\|\lambda\| = 0$ for the case $\lambda(t) = t$. The set Λ is like the one defined for the Skorokhod metric with the added proviso that $\|\lambda\|$ be finite; both λ and λ^{-1} must be strictly increasing functions. Then define

$$d_B(x,y) = \inf_{\lambda\in\Lambda}\left\{\varepsilon > 0\colon \|\lambda\| \leq \varepsilon,\ \sup_t |x(t)-y(\lambda(t))| \leq \varepsilon\right\}. \tag{28.14}$$

We review the essential properties of d_B.

28.6 Theorem d_B is a metric.

Proof $d_B(x,y) = 0$ iff $x = y$ is immediate. $d_B(x,y) = d_B(y,x)$ is also easy once it is noted that $\|\lambda^{-1}\| = \|\lambda\|$. To show the triangle inequality, note that

$$\begin{aligned}\|\lambda_1\| + \|\lambda_2\| &= \sup_{t\neq s,\,t'\neq s'}\left\{\left|\log\frac{\lambda_1(t')-\lambda_1(s')}{t'-s'}\right| + \left|\log\frac{\lambda_2(t)-\lambda_2(s)}{t-s}\right|\right\}\\ &\geq \sup_{t\neq s,\,t'\neq s'}\left\{\left|\log\frac{\big(\lambda_1(t')-\lambda_1(s')\big)\big(\lambda_2(t)-\lambda_2(s)\big)}{(t'-s')(t-s)}\right|\right\}\\ &\geq \sup_{t\neq s}\left\{\left|\log\frac{\lambda_1(\lambda_2(t))-\lambda_1(\lambda_2(s))}{t-s}\right|\right\}\end{aligned} \tag{28.15}$$

where in the last line we considered the case $t' = \lambda_2(t)$ and $s' = \lambda_2(s)$. Thus,

$$\|\lambda_1\| + \|\lambda_2\| \geq \|\lambda_1\circ\lambda_2\|, \tag{28.16}$$

where $\lambda_1\circ\lambda_2(t) = \lambda_1(\lambda_2(t))$, and $\lambda_1\circ\lambda_2$ is clearly an element of Λ. Since

$$\sup_t |x(t)-z(\lambda_1(\lambda_2(t)))| \leq \sup_t |x(t)-y(\lambda_2(t))| + \sup_t |y(t)-z(\lambda_1(t))| \tag{28.17}$$

by the ordinary triangle inequality for points of $\mathbb{R}$, the condition $d_S(x,z) \leq d_S(x,y) + d_S(y,z)$ follows from the definition. ∎

Next we explore the relationship between d_S and d_B, and verify that they are equivalent metrics. Inequalities going in both directions can be derived provided the distances are sufficiently small. Given functions x and y for which $d_B(x,y) =$