Choose $\varepsilon = (K^*)^{3/4}h^{3/2}$ to minimize the last member above, and we obtain

$$E|\Delta_{t+h,t}|^3 \leq 4(K^*)^{3/4}h^{3/2}. \tag{29.32}$$

This condition verifies (29.22), and completes the proof. ■

Notice how (29.30) is a substantial strengthening of the Chebyshev inequality, which gives merely $P(|\Delta_{t+h,t}| \geq \alpha) \leq h/\alpha^2$. We have not assumed the existence of the third moment at the outset; this emerges (along with the Gaussianity) from the assumption of independent increments of arbitrarily small width, which allows us to take (29.29) to the limit.

29.2 Asymptotic Independence

Let $\{X_n\}_1^\infty$ denote a stochastic sequence in $(D,\mathcal{B}_D)$. We say that X_n has asymptotically independent increments if, for any collection of points $\{s_i,t_i,\ i = 1,...,r\}$ such that

$$0 \leq s_1 \leq t_1 < s_2 \leq t_2 < \ldots < s_r \leq t_r \leq 1,$$

and all collections of linear Borel sets $B_1,...,B_r \in \mathcal{B}$,

$$\left| P\big(X_n(t_i) - X_n(s_i) \in B_i,\ i = 1,...,r\big) - \prod_{i=1}^{r} P\big(X_n(t_i) - X_n(s_i) \in B_i\big) \right| \longrightarrow 0 \tag{29.33}$$

as $n \longrightarrow \infty$. Notice that in this definition, gaps of positive width are allowed to separate the increments, which will be essential to establish asymptotic independence in the partial sums of mixing sequences. The gaps can be arbitrarily small, however, and continuity allows us to ignore them as we see below.

Given this idea, we have the following consequence of **29.1**.

29.4 Theorem Let $\{X_n\}_{n=1}^\infty$ have the following properties:

(a) The increments are asymptotically independent.

(b) For any $\varepsilon > 0$ and $\eta > 0$, $\exists\ \delta \in (0,1)$ s.t. $\limsup_{n\to\infty} P(w(X_n,\delta) \geq \varepsilon) \leq \eta$.

(c) $\{X_n(t)^2\}_{n=1}^\infty$ is uniformly integrable for each $t \in [0,1]$.

(d) $E(X_n(t)) \to 0$ and $E(X_n(t)^2) \to t$ as $n \to \infty$, each $t \in [0,1]$.

Then $X_n \xrightarrow{D} B$. □

Be careful to note that $w(.,\delta)$ in (b) is the modulus of continuity of (27.14), *not* w' of (28.3).

Proof Condition (b), and the fact that $E|X_n(0)| \longrightarrow 0$ by (d), imply by **28.14** that the associated sequence of p.m.s is uniformly tight. Theorem **26.22** then implies that the latter sequence is compact, and so has one or more cluster points. To complete the proof, we show that all such cluster points must have the characteristics of Wiener measure, and hence that the sequence has this p.m. as its unique weak limit.

Consider the properties the limiting p.m. must possess. Writing X for the random element, **28.14** also gives $P(X \in C) = 1$. Uniform integrability of $X_n(t)^2$, and hence of $X_n(t)$, implies that $E(X(t)) = 0$ and $E(X(t)^2) = t$, by **22.16**. By condition (a) we may say that the increments $X(t_1) - X(s_1),...,X(t_r) - X(s_r)$ are totally independent according to (29.33). Specifically, consider increments $X(t_i) - X(s_i)$ and $X(t_{i+1}) - X(s_{i+1})$ for the case where $s_{i+1} = t_i + 1/m$. By a.s. continuity,

$$\lim_{m\to\infty} \big(X(t_{i+1}) - X(t_i + 1/m)\big) = X(t_{i+1}) - X(t_i) \text{ w.p.1,} \tag{29.34}$$

so that asymptotic independence extends to contiguous increments. All the conditions of **29.1** are therefore satisfied by X, and $X \sim B$. ■

Our aim is now to get a FCLT for partial-sum processes by linking up the asymptotic independence idea with our established characterization of a dependent increment process; that is to say, as a near-epoch dependent function of a mixing process. Making this connection is perhaps the biggest difficulty we still have to surmount. An approach comparable to the 'blocking' argument used in the CLTs of §24.4 is needed; and in the present context we can proceed by mapping an infinite sequence into [0,1] and identifying the increments with asymptotically independent blocks of summands. This is a particularly elegant route to the result. However, an asymptotic martingale difference-type property of the type exploited in **24.6** is not going to work in our present approach to the problem. While the terms of a mixing process (of suitable size) can be 'blocked' so that the blocks are asymptotically independent (more or less by definition of mixing), mixingale theory will not serve here; near-epoch dependent functions can be dealt with only by a direct approximation argument.

What we shall show is that, if the difference between two stochastic processes is $o_p(1)$ and one of them exhibits asymptotic independence, so must the other, in a sense to be defined. Near-epoch dependent functions can be approximated in the required way by their near-epoch conditional expectations, where the latter are functions of mixing variables. This result is established in the following lemma in terms of the independence of a pair of sequences, which in the application will be adjacent increments of a partial sum process.

29.5 Lemma (Wooldridge and White 1986: Lemma A.3) If $\{Y_{jn}\}$ and $\{Z_{jn}\}$ are real stochastic sequences, and

(a) $Y_{jn} - Z_{jn} \xrightarrow{pr} 0$, for $j = 1,2$;

(b) $Y_{jn} \xrightarrow{D} Y_j$ for $j = 1,2$;

(c) for any $A_1, A_2 \in \mathcal{B}$,

$$|P(\{Z_{1n} \in A_1\} \cap \{Z_{2n} \in A_2\}) - P(Z_{1n} \in A_1)P(Z_{2n} \in A_2)| \to 0 \tag{29.35}$$

as $n \to \infty$;

then

$$P(\{Y_{1n} \in B_1\} \cap \{Y_{2n} \in B_2\}) \to P(Y_1 \in B_1)P(Y_2 \in B_2) \tag{29.36}$$

for all Y_j-continuity sets (sets $B_j \in \mathcal{B}$ such that $P(Y_j \in \partial B_j) = 0$) for $j = 1,2$.

Proof Considering (Z_{1n},Z_{2n}) and (Y_{1n},Y_{2n}) as points of $\mathbb{R}^2$ with the Euclidean metric, (a) implies $d_E\big((Z_{1n},Z_{2n}),(Y_{1n},Y_{2n})\big) \to 0$, and by an application of **26.24**, (b) implies both $(Z_{1n},Z_{2n}) \xrightarrow{D} (Y_1,Y_2)$, and $j = 1,2$. Write