exists a linear transformation $(\boldsymbol{D}'\boldsymbol{D})^{-1}\boldsymbol{D}'$, under which the elements of the limiting process are independent of one another.

As we did for the scalar case, we review some of the simplest sets of sufficient conditions. Let $\boldsymbol{U}_{ni} = \boldsymbol{S}_n^{-1/2}\boldsymbol{U}_i$ where $\boldsymbol{S}_n = E\big(\sum_{i=1}^n \boldsymbol{U}_i \sum_{i=1}^n \boldsymbol{U}_i'\big)$. For this choice, $\boldsymbol{D} = \boldsymbol{I}_m$ is imposed automatically. If $\{\boldsymbol{U}_i\}$ is uniformly L_r-bounded, choose $c_{ni}^{\lambda} = (\boldsymbol{\lambda}'\boldsymbol{S}_n^{-1}\boldsymbol{\lambda})^{1/2}$. Then $\boldsymbol{\lambda}'\boldsymbol{U}_{ni}/c_{ni}^{\lambda}$ is a linear combination of the $\boldsymbol{U}_i$ with weights summing to 1 and $\sup_{i,n}\|\boldsymbol{\lambda}'\boldsymbol{U}_{ni}/c_{ni}^{\lambda}\|_r < \infty$ holds for any $\boldsymbol{\lambda}$, so that conditions (a) and (b) of **29.6** are satisfied. The multivariate analogue of **29.7** is then easily obtained:

29.19 Corollary Let $\{\boldsymbol{U}_i\}$ be a zero-mean, uniformly L_r-bounded m-vector sequence, with each element L_2-NED of size $-\frac{1}{2}$ on an α-mixing process of size $-r/(r-2)$; and assume $n^{-1}\boldsymbol{S}_n \to \boldsymbol{\Omega} < \infty$. If $\boldsymbol{X}_n(t) = n^{-1/2}\sum_{i=1}^{[nt]}\boldsymbol{U}_i$, then $\boldsymbol{X}_n \xrightarrow{D} \boldsymbol{B}(\boldsymbol{\Omega})$. □

Compare this formulation with (27.82), and as with **29.7**, note the important difference from the martingale difference case, with $\boldsymbol{\Omega}$ taking the place of $\boldsymbol{\Sigma}$. It is also worth reiterating how the statement of conditions is potentially misleading, given that the last one is typically hard to fulfil without a NED size of -1.

Somewhat trickier is the case of trending moments, where different elements of the vector may even be subject to different trends. The discussion here will have some close parallels with §24.4. Diagonalize $\boldsymbol{S}_n$ as

$$\boldsymbol{S}_n = \boldsymbol{C}_n\boldsymbol{M}_n\boldsymbol{C}_n', \tag{29.72}$$

where $\boldsymbol{M}_n$ is diagonal of rank m, and $\boldsymbol{C}_n\boldsymbol{C}_n' = \boldsymbol{C}_n'\boldsymbol{C}_n = \boldsymbol{I}_m$. Assume, to fix ideas, that $\boldsymbol{C}_n \to \boldsymbol{C}$, which can be thought of as imposing a form of global stationarity on the cross-correlations. Then $\boldsymbol{S}_n^{-1/2} = \boldsymbol{M}_n^{-1/2}\boldsymbol{C}_n'$ and

$$E\big(\boldsymbol{X}_n(t)\boldsymbol{X}_n(t)'\big) = \boldsymbol{M}_n^{-1/2}\boldsymbol{C}_n'E\left(\sum_{i=1}^{[nt]}\boldsymbol{U}_i\sum_{i=1}^{[nt]}\boldsymbol{U}_i'\right)\boldsymbol{C}_n\boldsymbol{M}_n^{-1/2}$$

$$\approx \boldsymbol{M}_n^{-1/2}\boldsymbol{M}_{[nt]}\boldsymbol{M}_n^{-1/2} \to \boldsymbol{H}(t), \tag{29.73}$$

where the approximation is got by setting $\boldsymbol{C}_n$ to $\boldsymbol{C}$, and can be made as good as desired by taking n large enough. The status of conditions **29.18**(a) and (b) must be checked by evaluating the elements of $\boldsymbol{H}$ in (29.73). An example is the best way to illustrate the possibilities.

29.20 Example Let $m = 2$, and assume $E(\boldsymbol{U}_i\boldsymbol{U}_j') = \boldsymbol{0}$ for $i \neq j$, but let

$$E(\boldsymbol{U}_i\boldsymbol{U}_i') = \boldsymbol{C}\begin{bmatrix} i^{\beta_1} & 0 \\ 0 & i^{\beta_2}\end{bmatrix}\boldsymbol{C}' \tag{29.74}$$

for fixed $\boldsymbol{C}$. Then, $\boldsymbol{M}_n = \mathrm{diag}\{n^{\beta_1+1}/(\beta_1+1),\ n^{\beta_2+1}/(\beta_2+1)\}$ and $\boldsymbol{H}(t) = \mathrm{diag}\{t^{\beta_1+1}, t^{\beta_2+1}\}$. For $\beta_1, \beta_2 > -1$, t^{β_1+1} and t^{β_2+1} are increasing homeomorphisms on the unit square, and condition **29.18**(b) is satisfied. It remains to check **29.18**(a). Condition **29.14**(f′) holds for the array $\{\boldsymbol{\lambda}'\boldsymbol{U}_{ni}\}$ with respect to

$$\eta^{\lambda}(t) = \lambda_1^2 t^{\beta_1+1} + \lambda_2^2 t^{\beta_2+1}, \tag{29.75}$$

which, since $\lambda_1^2 + \lambda_2^2 = 1$, is an increasing homeomorphism on the unit square with $\eta^\lambda(1) = 1$ and $\eta^\lambda(0) = 0$ whenever $\beta_1, \beta_2 > -1$. Assuming that **29.6**(b) holds for

$$c_{ni}^\lambda = \|\boldsymbol{\lambda}'\boldsymbol{U}_{ni}\|_2 = \left[\lambda_1^2\left(\frac{(\beta_1+1)i^{\beta_1}}{n^{\beta_1+1}}\right) + \lambda_2^2\left(\frac{(\beta_2+1)i^{\beta_2}}{n^{\beta_2+1}}\right)\right]^{1/2}, \tag{29.76}$$

we can check conditions **29.6**(d) and **29.6**(e). The latter holds for $\gamma = \frac{1}{2}$. We also find that

$$\begin{aligned} \frac{\nu_n^2(t,\delta)}{\delta} &= \frac{1}{\delta}\sum_{i=[nt]+1}^{[n(t+\delta)]}(c_{ni}^\lambda)^2 \\ &\approx \frac{\lambda_1^2\big((t+\delta)^{\beta_1+1} - t^{\beta_1+1}\big) + \lambda_2^2\big((t+\delta)^{\beta_2+1} - t^{\beta_2+1}\big)}{\delta} \\ &\to \lambda_1^2(\beta_1+1)t^{\beta_1} + \lambda_2^2(\beta_2+1)t^{\beta_2} \end{aligned} \tag{29.77}$$

as $\delta \to 0$, where the approximation is as good as desired with large enough n. Condition **29.6**(d) is therefore satisfied, and hence **29.18**(a) holds, for the cases where $\beta_1, \beta_2 \geq 0$. This completes the verification of the conditions. □