ω, this Riemann-Stieljtes integral *does not exist*; quite simply, we have not required $M(\omega,.)$ to be of bounded variation, and the example of Brownian motion shows that this requirement could fail for almost all ω. Hence, a different interpretation of the process $I(t)$ is called for.

The results we shall obtain are actually available for a larger class of integrator functions, including martingales whose quadratic variation is a stochastic process. However, it is substantially easier to prove the existence of the integral for the case indicated, and this covers the applications of interest to us.

We assume the existence of a filtration $\{\mathcal{F}(t),\ t \in [0,1]\}$, on a probability space $(\Omega,\mathcal{F},P)$. Let

$$\alpha\colon [0,1] \mapsto \mathbb{R}$$

be a positive, increasing element of D, and let $\alpha(0) = 0$ and $\alpha(1) = 1$, with no loss of generality as it turns out. For any $t \in (0,1]$, the restriction of α to $[0,t]$ induces a finite Lebesgue-Stieltjes measure. That is to say, α is a c.d.f., and the function $\int_B d\alpha(s)$ assigns a measure to each $B \in \mathcal{B}_{[0,t]}$. Accordingly we can define on the product space $(\Omega\times[0,t],\ \mathcal{F}(t)\otimes\mathcal{B}_{[0,t]})$ the product measure μ_α, where

$$\mu_\alpha(A) = \int_\Omega\int_0^t 1_A(\omega,s)d\alpha(s)dP(\omega) = E\left(\int_0^t 1_A(\omega,s)d\alpha(s)\right) \tag{30.36}$$

for each $A \in \mathcal{F}(t)\otimes\mathcal{B}_{[0,t]}$.

Let $\mathbb{L}_\alpha$ denote the class of functions

$$f\colon \Omega \mapsto R_{[0,1]}$$

which are (a) progressively measurable (and hence adapted to $\{\mathcal{F}(t)\}$), and (b) square-integrable with respect to μ_α; that is to say, $\|f\| < \infty$ where

$$\|f\| = E\left(\int_0^1 f^2 d\alpha\right)^{1/2}. \tag{30.37}$$

It is then easy to verify that $\|f-g\|$ is a pseudo-metric on $\mathbb{L}_\alpha$. While $\|f-g\| = 0$ does not guarantee that $f(\omega) = g(\omega)$ for every $\omega \in \Omega$, it does imply that the integrals of f and g with respect to α will be equal almost surely $[P]$. In this case we call functions f and g *equivalent*.

The chief technical result we need is to show that a class of simple functions is dense in $\mathbb{L}_\alpha$. Let $\mathbb{E}_\alpha \subseteq \mathbb{L}_\alpha$ denote the class such that $f(t) = f(t_k)$ for $t \in [t_k,t_{k+1})$, $k = 0,\dots,m-1$ and $f(1) = f(1-)$, where $\{t_1,\dots,t_m\} = \Pi_m$ is a partition of $[0,1]$ for some $m \in \mathbb{N}$.

30.9 Lemma (after Kopp 1984) For each $f \in \mathbb{L}_\alpha$, there exists a sequence $\{f_{(n)} \in \mathbb{E}_\alpha,\ n \in \mathbb{N}\}$ with $\|f_{(n)} - f\| \to 0$ as $n \to \infty$.

Proof Let the domain of f be extended to $\mathbb{R}$ by setting $f(t) = 0$ for $t \notin [0,1]$. By square-integrability, $\int_{-\infty}^{+\infty} f(\omega,t)^2 d\alpha(t) < \infty$ a.s.$[P]$, and

$$\int_{-\infty}^{+\infty}\big(f(\omega,t+h) - f(\omega,t)\big)^2 d\alpha(t) \to 0 \text{ a.s.}[P] \text{ as } h \to 0;$$

hence